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J. Phys. A: Math. Gen. **30** (1997) 4731–4755. Printed in the UK PII: S0305-4470(97)79012-7

Macroscopic quantum systems as measuring devices: dc SQUIDs and superselection rules

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Received 28 October 1996, in final form 1 April 1997

Abstract. Using a Hilbert space formalism we present axiomatic models of both a current-fed thick superconducting ring and a dc SQUID (superconducting quantum interference device) as quantum systems possessing superselection rules. A method of *quantization by parts* is introduced to establish a quantum theory of a system having a circuit configuration. This involves separate quantization of parts of a circuit: the whole system is then recovered by adding these separately quantized parts together. Our models make clear the difference between standard quantum interference and the interference effects exhibited by SQUIDs. They lead us to question a commonly accepted definition of a classical system, and also clarify the properties required of measuring apparatus in the quantum and classical realms.

1. Introduction

Superselection rules (SSRs), and the associated concept of macrorealism, provide a well established means of describing classical properties within a quantum formalism [1, 2]. To date, they have been most extensively employed in quantum measurement theory [3–5], but they might be expected to be a general feature of all quasiquantum systems (those exhibiting both quantum and classical aspects). To support this contention, specific examples of systems which can be axiomatically modelled using a quantum mechanics plus SSR formalism are needed. Chiral molecules are widely regarded as one such example [6–8]. In a recent paper, we presented two models of quantum systems possessing SSRs; namely a thick superconducting ring (hereafter abbreviated to TSCR) and a TSCR containing a single Josephson junction (JJ) [9]. In addition to providing an application for SSRs outside quantum measurement theory, these models help shed some light on the non-observable nature of the generator of a unitary time evolution describing transitions between supersectors [1, 3, 5].

In this paper, we use similar methods to develop an axiomatic model of the phenomenology of a current-fed TSCR, and then a model of a TSCR containing two JJs in a dc SQUID (superconducting quantum interference device) configuration, in which the JJs are parallel with respect to the fed current passing through the ring. In the presence of an externally applied magnetic flux $\Phi_{\rm ex}$, the maximum through supercurrent that the system can sustain displays an interference pattern as a function of $\Phi_{\rm ex}$, with a period equal to the flux quantum $\Phi_0 = h/2e$. This extreme sensitivity to the external magnetic flux makes the use of SQUIDs ideal in the construction of high-resolution magnetometers and related measuring apparatus [10, 11]. Our models provide further examples of physical systems exhibiting SSRs but are also of interest in their own right. They clarify the difference between standard quantum interference patterns and the interference effects exhibited by these systems. They also suggest that a commonly accepted definition of a classical system [12] needs amending, and help us to distinguish the essential features of apparatus used to measure quantum, and classical, systems.

There is a very well respected microscopic theory of superconductivity: the BCS theory [13], but a large number of superconducting phenomena may be modelled successfully using the macroscopic wavefunction approach [14, 15]. According to BCS theory, superconductivity becomes possible because at low temperatures the electrons form pairs, known as Cooper pairs, which behave as bosons and thus can all occupy the ground-state energy level, forming a Bose–Einstein condensate. It is the flow of this condensate as a whole that gives rise to a supercurrent. The macroscopic wavefunction hypothesis assumes that the whole condensate may be treated as a single pseudoparticle of mass $m = 2m_e$ and charge $q = 2e$ twice those of an electron. This pseudoparticle can then be described by a one-particle wavefunction. We shall adopt the macroscopic wavefunction approach in this paper and shall use the method of quantization of classical particle models to construct our quantum mechanical models. Since we shall be considering a circuit configuration, a method of *quantization by parts* is introduced. This involves separate quantization of the parts of a circuit: the whole system is then recovered by adding these separately quantized parts together.

2. A TSCR subject to an applied current

The first system that we wish to study is a uniform TSCR under standard conditions [9], subject to a current flowing in at the top ($\theta = 0$), passing in parallel through the left- and right-hand sides of the ring and recombining in the output lead situated at $\theta = \theta_0$, where θ is taken in an anticlockwise direction (figure 1). The left-hand segment thus subtends an angle θ_0 and the right-hand segment subtends an angle $(2\pi - \theta_0)$. For a uniform ring of total self-inductance L, the self-inductances of these segments are respectively L_l and L_r:

$$L_l = \frac{\theta_0}{2\pi} L \qquad L_r = \frac{2\pi - \theta_0}{2\pi} L. \tag{1}$$

In a resistanceless parallel circuit in which the mutual inductance between the paths is negligible the current in each path is inversely proportional to the self-inductances of

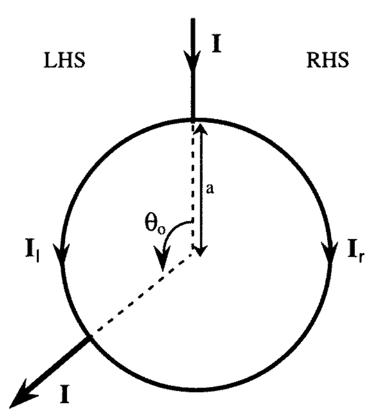

Figure 1. TSCR subject to an applied current.

the two paths [16 p 13]. A supercurrent I fed into the TSCR will split into left and right currents I_l and I_r:

$$I_l = \frac{(2\pi - \theta_0)}{2\pi} I \qquad I_r = \frac{\theta_0}{2\pi} I. \tag{2}$$

The currents I_l and I_r each produce a flux of magnitude Φ_I within the ring in opposing directions, so that the net flux enclosed in the ring is zero. The magnitude of Φ_I is determined by I and θ_0:

$$\Phi_I = L_l I_l = L_r I_r = \frac{(2\pi - \theta_0)\theta_0}{(2\pi)^2} L I. \tag{3}$$

The superconductivity of a material is destroyed by currents (or, equally, external fields) above some critical value J_c (or B_c) [16]. This means that for certain values of I, equation (2) places upper and lower limits on θ_0, since both the left and right currents must remain below the critical current J_c in order to maintain superconductivity. As the output lead gets closer to the input one (i.e. as either $\theta_0 \to 0$ or $\theta_0 \to 2\pi$), the splitting of the currents between the left- and right-paths will become increasingly unequal so that most of the applied current I will flow through one path. In the limiting case superconductivity will be destroyed for values of the applied current exceeding the critical current, whereas if the input and output leads are opposite each other (so that $I_r = I_l = I/2$) the applied current can be twice the critical current before superconductivity is destroyed.

2.1. Modelling the system in the absence of magnetic fields

In our previous paper [9], we developed quantum models by considering a single classical particle of mass $m = 2m_e$ and charge $q = 2e$ twice those of an electron, constrained to move around a circle $\mathcal{S}$ of radius a, and then quantizing this classical model. In the present case, if we were to start with a classical model we would have a particle moving *through* the ring: being classical it would have to take either one or other of the two paths. This is typical of an electrical circuit. There have been some systematic investigations into quantum systems constrained in both similar, and more complex, circuits in which intricate mathematical analysis has to be used [17, 18]. Fortunately the system that we are considering here turns out to be mathematically less complex, and exactly solvable.

We start with a classical system consisting of a single particle moving through the ring, taking either one or other of the two paths. Each of these paths will be treated separately with its own position variable and associated canonical momentum. We take the ring to be lying in the x–y plane centred at the origin, and shall treat the system as one-dimensional. In cylindrical coordinates, the position variable for the left (or right) path is $a\theta$ (or $a(2\pi - \theta)$) and the momentum p_l (or p_r). The Hamiltonian (energy) of the system is taken to be the weighted average of the energies of the two possible classical paths, with the relative weights determined by the path length ratio $\theta_0 : (2\pi - \theta_0)$. That is, we have

$$H = \frac{1}{2m}\left[\frac{\theta_0}{2\pi} p_l^2 + \left(\frac{2\pi - \theta_0}{2\pi}\right) p_r^2\right]. \tag{4}$$

We are now in a position to implement a quantization process. We first quantize the motion on the left- and right-hand sides in separate Hilbert spaces $\mathcal{H}_l$ and $\mathcal{H}_r$, with separate left and right observables. We then associate the direct sum Hilbert space $\mathcal{H} = \mathcal{H}_l \oplus \mathcal{H}_r$, and the direct sums of the separately quantized operators, with the entire system [9, 17, 18]. This will lead us to introduce further operators representing observables of the whole ring, such as through and circulating momenta. That we have chosen to represent the entire system by a direct sum rather than a tensor product [19] is a reflection of the fact that we are dealing with a single system, not two coupled systems. Note that no SSRs are implied by the direct sums here.

2.1.1. Quantization of the two sides. To quantize the left-hand side, we take the Hilbert space $\mathcal{H}_l$ to be the space $L^2(\mathcal{S}_l)$ of square-integrable functions on the left-hand segment of the ring $\mathcal{S}_l = \{\theta \in [0, \theta_0]\}$ with respect to the measure $a\,\mathrm{d}\theta$. Let $AC(\mathcal{S}_l)$ denote the set of absolutely continuous functions $\phi^l(\theta)$ on $\mathcal{S}_l$ and let φ_l denote a real number in the interval $(-\pi, \pi]$. Then the operator $\hat{p}_{\varphi_l} = -(\mathrm{i}\hbar/a)(\mathrm{d}/\mathrm{d}\theta)$ will be self-adjoint [20] on the domain

$$D_{\varphi_l} = \{\phi^l : \phi^l \in AC(\mathcal{S}_l);\ \phi^l(0_+) = \mathrm{e}^{\mathrm{i}\varphi_l}\phi^l(\theta_{0-});\ \hat{p}_{\varphi_l}\phi^l \in L^2(\mathcal{S}_l)\}. \tag{5}$$

We make the assumption that the canonical momentum is quantized as the operator $\hat{p}_{\varphi_l}$. The phase parameter φ_l is determined by θ_0 and I, as will be seen later. As in our earlier models [9], we can define a left-hand current operator $\hat{J}_{\varphi_l}$ representing the supercurrent in the left-hand path:

$$\hat{J}_{\varphi_l} = \frac{e}{\pi a m}\hat{p}_{\varphi_l}. \tag{6}$$

We can also introduce a *generated flux* operator $\hat{\Phi}_{\varphi_l}$ representing the flux generated in the ring by the current $\hat{J}_{\varphi_l}$ flowing in the left-hand side of the ring:

$$\hat{\Phi}_{\varphi_l} = L_l\hat{J}_{\varphi_l} = \frac{\theta_0 L}{2\pi}\frac{e}{\pi a m}\hat{p}_{\varphi_l} = \frac{\theta_0}{2\pi}\frac{\pi a}{e}\hat{p}_{\varphi_l} \tag{7}$$

where L is taken to be $L = m(\pi a/e)^2$, as in our previous paper [9]. These operators share a common set of eigenfunctions

$$\psi^l_{\varphi_l,k_l}(\theta) = \frac{1}{\sqrt{2\pi a}}\exp\mathrm{i}\left[\frac{2\pi k_l - \varphi_l}{\theta_0}\right]\theta \qquad k_l = 0, \pm 1, \pm 2, \ldots. \tag{8}$$

These eigenfunctions are not normalized: the factor $1/\sqrt{2\pi a}$ has been chosen so as to ensure that the condensate density will be uniform and normalized round the entire ring [21]. Respective eigenvalues are:

$$p_{\varphi_l,k_l} = \frac{\hbar}{a\theta_0}(2\pi k_l - \varphi_l) \qquad j_{\varphi_l,k_l} = \frac{\Phi_0}{L\theta_0}(2\pi k_l - \varphi_l) \qquad \Phi_{\varphi_l,k_l} = \frac{\Phi_0}{2\pi}(2\pi k_l - \varphi_l). \tag{9}$$

We need to determine the value of the phase parameter φ_l. We have already defined the left (and right) generated flux Φ_I in terms of the external parameters I, L and θ_0 (equation (3)). By equating Φ_I with the eigenvalues of the left generated flux operator $\hat{\Phi}_{\varphi_l}$ (equation (9)), we can write φ_l and k_l in terms of Φ_I:

$$\Phi_I = \Phi_{\varphi_l,k_l} = \frac{\Phi_0}{2\pi}(2\pi k_l - \varphi_l) \Longrightarrow k_l - \frac{\varphi_l}{2\pi} = \frac{\Phi_I}{\Phi_0}. \tag{10}$$

The quantity Φ_I/Φ_0 may be uniquely expressed as the sum of an integer N and a remainder α:

$$\frac{\Phi_I}{\Phi_0} = N + \alpha \qquad -\tfrac{1}{2} \leqslant \alpha < \tfrac{1}{2}. \tag{11}$$

As k_l is an integer and $-\frac{1}{2} \leqslant -\varphi_l/2\pi < \frac{1}{2}$, it follows that $N = k_l$ and $\alpha = -\varphi_l/2\pi$. The values of both the parameters φ_l and k_l are completely determined by the values of the external parameters I and θ_0, which also determine Φ_I. Here we have an interesting situation in that the external physical parameters (I, θ_0) determine not only a unique eigenfunction and hence a unique state for the system, but also fix the phase parameter φ_l which defines the momentum operator $\hat{p}_{\varphi_l}$. Different values of (I, θ_0) define different momentum operators: there is a sense in which TSCRs with different (I, θ_0) are different physical systems. We can highlight the dependence of the state on (I, θ_0) by labelling the eigenfunctions $\psi^l_{\varphi_l,k_l}(\theta)$ as $\psi^l_{I,\theta_0}(\theta)$:

$$\psi^l_{\varphi_l,k_l}(\theta) = \psi^l_{I,\theta_0}(\theta) = \frac{1}{\sqrt{2\pi a}} \exp \mathrm{i} \left(\frac{2\pi \Phi_I}{\theta_0 \Phi_0} \right) \theta. \tag{12}$$

Likewise, the eigenvalues may be labelled as

$$p_{\varphi_l,k_l} = p^l_{I,\theta_0} = \frac{2e\Phi_I}{a\theta_0} \qquad j_{\varphi_l,k_l} = j^l_{I,\theta_0} = \frac{\Phi_I}{L_l} = I_l \qquad \Phi_{\varphi_l,k_l} = \Phi^l_{I,\theta_0} = \Phi_I. \tag{13}$$

The right-hand side is quantized in the same manner. We identify $\mathcal{H}_r$ with the Hilbert space $L^2(\mathcal{S}_r)$ of square-integrable functions on the interval $\mathcal{S}_r = \{\theta \in [\theta_0, 2\pi]\}$. We define the domain and the operators $\hat{p}_{\varphi_r}$ (representing momentum), $\hat{J}_{\varphi_r}$ (current) and $\hat{\Phi}_{\varphi_r}$ (flux) by analogy with the left-hand side. Since φ_r and the quantum number k_r which labels the common eigenfunctions of these operators are found to be fully determined by the external parameters (I, θ_0), so that $k_l = k_r = k$ and $\varphi_l = \varphi_r = \varphi$, we shall drop the subscripts from now on whenever no confusion is likely. The right-hand eigenfunctions of momentum, current and flux are

$$\psi^r_{\varphi_r,k_r}(\theta) = \psi^r_{I,\theta_0}(\theta) = \frac{1}{\sqrt{2\pi a}} \exp \mathrm{i} \left[\frac{2\pi \Phi_I}{(2\pi - \theta_0)\Phi_0} (2\pi - \theta) \right] \tag{14}$$

and their right-hand eigenvalues are

$$p^r_{I,\theta_0} = -\frac{2e\Phi_I}{a(2\pi - \theta_0)} \qquad j^r_{I,\theta_0} = -\frac{\Phi_I}{L_r} = -I_r \qquad \Phi^r_{I,\theta_0} = -\Phi_I. \tag{15}$$

The right-hand current eigenvalues j^r_{I,θ_0} are negative because the current is actually flowing down through the right-hand side, not up around it.

Note that the cancelling out of left and right generated fluxes implies that $\psi^l_{I,\theta_0}(\theta_0) = \psi^r_{I,\theta_0}(\theta_0)$. This, together with the fact that $\psi^l_{I,\theta_0}(0) = \psi^r_{I,\theta_0}(2\pi)$, ensures that the wavefunction is single valued at the points $\theta = 0$ and $\theta = \theta_0$. As will be seen later, this enables us to introduce a continuous wavefunction around the entire ring.

2.1.2. Quantization of the entire ring. To quantize the entire ring, we take the following steps.

(1) We associate the direct sum Hilbert space $\mathcal{H} = \mathcal{H}_l \oplus \mathcal{H}_r$ with the entire ring. State vectors (wavefunctions) of the system must be of the form $\phi^l \oplus \phi^r$ with $\phi^l \neq 0 \neq \phi^r$, since $\phi^l = 0$, for example, would imply the condensate being solely on the right-hand side, which is never the case.

(2) Observables pertaining to only one side are represented by the extension of their operators in one or other of the subspaces $\mathcal{H}_l$ and $\mathcal{H}_r$ to operators in the direct sum space $\mathcal{H}$. These extensions must satisfy two requirements:

(a) they must possess the same eigenvalues as the original operators;

(b) their eigenfunctions must be state vectors of the system; that is, not of the form $\phi^l \oplus 0$ or $0 \oplus \phi^r$.

In the present case, for any given (I, θ_0), there exists a unique state for each side of the ring, so that there is only one possible combination of left- and right-hand states representing a pure state of the system. The states of the ring are

$$\Psi_{I,\theta_0} = \Psi_{\varphi,k} = \psi^l_{\varphi,k} \oplus \psi^r_{\varphi,k}. \tag{16}$$

The operator extensions must admit these state vectors as eigenvectors. This helps to establish how the operators should be extended. The apparently natural expressions $\hat{p}_{\varphi_l} \oplus \hat{\mathbb{I}}_r$ and $\hat{\mathbb{I}}_l \oplus \hat{p}_{\varphi_r}$, or $\hat{p}_{\varphi_l} \oplus \hat{0}_r$ and $\hat{0}_l \oplus \hat{p}_{\varphi_r}$, are not an appropriate choice since these operators do not satisfy the above two requirements.

(3) Instead, the left- and right-hand canonical momentum operators are represented in $\mathcal{H}$ by the operator extensions:

$$\hat{P}^{l}_{\varphi} = \hat{p}_{\varphi_l} \oplus C^{l}_{I,\theta_0}\hat{\mathbb{I}}_r \qquad \text{and} \qquad \hat{P}^{r}_{\varphi} = C^{r}_{I,\theta_0}\hat{\mathbb{I}}_l \oplus \hat{p}_{\varphi_r} \qquad (\varphi = \varphi_l = \varphi_r) \tag{17}$$

where $\hat{\mathbb{I}}_l$ and $\hat{\mathbb{I}}_r$ are the identity operators in $\mathcal{H}_l$ and $\mathcal{H}_r$ respectively. C^{l}_{I,θ_0} and C^{r}_{I,θ_0} are two constants determined by the external parameters (I, θ_0):

$$C^{l}_{I,\theta_0} = \frac{2e\Phi_I}{a\theta_0} \qquad C^{r}_{I,\theta_0} = -\frac{2e\Phi_I}{a(2\pi - \theta_0)}. \tag{18}$$

The operators $\hat{P}^{l}_{\varphi}$ and $\hat{P}^{r}_{\varphi}$ admit the state vectors Ψ_{I,θ_0} as eigenfunctions, since C^{l}_{I,θ_0} and C^{r}_{I,θ_0} are in fact the eigenvalues of $\hat{p}_{\varphi_l}$ and $\hat{p}_{\varphi_r}$ for any given (I, θ_0). It follows that the eigenvalues of $\hat{P}^{l}_{\varphi} + \hat{P}^{r}_{\varphi}$ and $\hat{P}^{l}_{\varphi} - \hat{P}^{r}_{\varphi}$ are respectively the sum and difference of the eigenvalues of $\hat{p}_{\varphi_l}$ and $\hat{p}_{\varphi_r}$.

Other extensions follow naturally. We have

$$\hat{J}^{l}_{\varphi} = \frac{e}{\pi am}\hat{P}^{l}_{\varphi} \qquad \hat{J}^{r}_{\varphi} = \frac{e}{\pi am}\hat{P}^{r}_{\varphi} \qquad \hat{\Phi}^{l}_{\varphi} = L_l\hat{J}^{l}_{\varphi} \qquad \hat{\Phi}^{r}_{\varphi} = L_r\hat{J}^{r}_{\varphi}. \tag{19}$$

The form of these operator extensions is a general feature of our quantization by parts approach. The reason why we cannot extend an operator simply by taking its direct sum with an identity operator is because we are dealing with a system (the condensate) which is always present on both sides of the ring, rather than a system which could be confined entirely to one subspace or the other.

(4) The Hamiltonian for the whole system (equation (4)) is quantized as the operator

$$\hat{H}_{\varphi} = \frac{1}{2m}\left[\frac{\theta_0}{2\pi}(\hat{P}^{l}_{\varphi})^2 + \left(\frac{2\pi - \theta_0}{2\pi}\right)(\hat{P}^{r}_{\varphi})^2\right] \tag{20}$$

which admits $\Psi_{I,\theta_0}(\theta)$ as an eigenfunction.

(5) We can define a *through momentum* operator $\hat{P}^{t}_{\varphi}$, representing the net momentum of the outgoing condensate:

$$\hat{P}^{t}_{\varphi} = \hat{P}^{l}_{\varphi} - \hat{P}^{r}_{\varphi}. \tag{21}$$

(6) The through current, whose eigenvalues we expect to be I for all θ_0, is represented by the operator

$$\hat{J}^{t}_{\varphi} = \hat{J}^{l}_{\varphi_l} + \hat{J}^{r}_{\varphi_r} \qquad \text{or equivalently} \qquad \hat{J}^{t}_{\varphi} = \frac{e}{\pi am}\hat{P}^{t}_{\varphi}. \tag{22}$$

(7) We can formally define a total enclosed flux operator $\hat{\Phi}_{T_\varphi}$ by considering the sum of the fluxes generated by the left- and right-hand currents.

$$\hat{\Phi}_{T_\varphi} = \hat{\Phi}^{l}_{\varphi} + \hat{\Phi}^{r}_{\varphi}. \tag{23}$$

Since there are no external magnetic fields present, we expect it to have a single eigenvalue of zero for all values of (I, θ_0).

All these operators share the same eigenfunctions as the Hamiltonian. Respective eigenvalues are:

$$\begin{aligned} &p^{l}_{I,\theta_0} = \frac{2e\Phi_I}{a\theta_0} \qquad p^{r}_{I,\theta_0} = -\frac{2e\Phi_I}{a(2\pi - \theta_0)} \qquad p^{t}_{I,\theta_0} = \frac{\pi am}{e}I \\ &j_{t_{I,\theta_0}} = I_l + I_r = I \qquad E_{I,\theta_0} = \frac{\Phi_I^2}{2L_l} + \frac{\Phi_I^2}{2L_r} \qquad \Phi_{T_{I,\theta_0}} = \Phi_I - \Phi_I = 0. \end{aligned} \tag{24}$$

As a function on the entire ring, the wavefunction is single valued and continuous at $\theta = 0$ and $\theta = \theta_0$. We may thus write it as

$$\Psi_{I,\theta_0}(\theta) = \chi_{\rm l}(\theta)\psi^{\rm l}_{I,\theta_0}(\theta) + \chi_{\rm r}(\theta)\psi^{\rm r}_{I,\theta_0}(\theta) \tag{25}$$

where the characteristic functions $\chi_{\rm l}(\theta)$ of the interval $\theta \in (0, \theta_0]$, and $\chi_{\rm r}(\theta)$ of the interval $\theta \in (\theta_0, 2\pi]$, have been introduced to convert functions originally defined on one or other of the subspaces $\mathcal{H}_{\rm l}$ or $\mathcal{H}_{\rm r}$ to elements of $\mathcal{H}$. The semiclosed intervals are necessary here to avoid 'doubling' the value of the function at $\theta = 0$ and $\theta = \theta_0$. This condensate wavefunction is continuous around the ring, with the left- and right-hand components always in phase at their meeting points. The importance of this feature will become apparent later when an external magnetic field is taken into account.

As anticipated, the enclosed flux is always zero and both definitions of the through current yield the applied current I as the eigenvalues. The energy eigenvalue is the sum of the stored magnetic energies associated with the flux Φ_I generated in each side of the ring.

2.1.3. Superselection rules. What we have here is a model of a one-state quantum system, similar to the case of an isolated TSCR studied in our previous paper [9]. The external parameters (I, θ_0) determine not only the operators $\hat{P}^{\rm l}_\varphi$ and $\hat{P}^{\rm r}_\varphi$ but also their specific eigenvalues $p^{\rm l}_{I,\theta_0}$ and $p^{\rm r}_{I,\theta_0}$, thus completely fixing the state of the system. There is no possibility of a coherent superposition of different states in $\mathcal{H}$ since there is no superposition of values of I or θ_0. In view of our subsequent studies, our aim here is to establish a model which will accommodate all possible values of the external parameters (I, θ_0) without allowing their coherent superposition. This may be achieved by forming direct integral Hilbert spaces and introducing SSRs.

We first construct the direct integral Hilbert spaces. Let $\mathcal{H}(I, \theta_0)$ be the subspace of $\mathcal{H}$ spanned by $\Psi_{I,\theta_0}(\theta)$. For any particular value of θ_0 we can construct a direct integral over the measure dI [22]:

$$\mathcal{H}^{\oplus}(\theta_0) = \int^{\oplus} \mathcal{H}(I, \theta_0)\, {\rm d}I \tag{26}$$

where the integral is over the range of values of I for which the system remains superconducting. The upper limit on I is determined by the requirement that neither of the currents in the left- and right-hand paths exceeds the critical current $J_{\rm c}$ of the bulk material. That is, both $I_{\rm l} < J_{\rm c}$ and $I_{\rm r} < J_{\rm c}$:

$$\Longrightarrow I < \frac{2\pi}{2\pi - \theta_0} J_{\rm c} \qquad \text{and} \qquad I < \frac{2\pi}{\theta_0} J_{\rm c}. \tag{27}$$

The upper bound $I^+_{\theta_0}$ on I is determined by whichever of these inequalities is the least. The direct integral over I is thus over the range of allowed values of I: $I \in (0, I^+_{\theta_0})$.

To accommodate different values of θ_0, we now construct a direct integral of the family of Hilbert spaces $\mathcal{H}^{\oplus}(\theta_0)$ with respect to the measure dθ_0:

$$\mathcal{H}^{\oplus} = \int^{\oplus} \mathcal{H}^{\oplus}(\theta_0)\, {\rm d}\theta_0 = \int^{\oplus} {\rm d}\theta_0 \int^{\oplus} \mathcal{H}(I, \theta_0)\, {\rm d}I \tag{28}$$

where the double direct integral is over the ranges $\theta_0 \in (0, 2\pi)$ and $I \in (0, I^+_{\theta_0})$.

Any particular (I, θ_0) determines φ and thus fixes the relevant set of operators $\hat{P}^{\rm l}_\varphi$, $\hat{P}^{\rm r}_\varphi$, $\hat{P}^{\rm t}_\varphi$, $\hat{J}^{\rm t}_\varphi$, $\hat{H}_\varphi$ and $\hat{\Phi}_{T_\varphi}$ in $\mathcal{H}$. Let $\hat{P}^{\rm l}(I, \theta_0)$, $\hat{P}^{\rm r}(I, \theta_0)$, $\hat{P}^{\rm t}(I, \theta_0)$, $\hat{J}^{\rm t}(I, \theta_0)$, $\hat{H}(I, \theta_0)$ and

$\hat{\Phi}_T(I, \theta_0)$ be their respective reductions to the subspace $\mathcal{H}(I, \theta_0)$. Then we can define direct integral operators acting on $\mathcal{H}^{\oplus}$as follows

$$\hat{P}_1^{\oplus} = \int^{\oplus} \mathrm{d}\theta_0 \int^{\oplus} \mathrm{d}I \; \hat{P}^1(I, \theta_0) \qquad \hat{\Phi}_T^{\oplus} = \int^{\oplus} \mathrm{d}\theta_0 \int^{\oplus} \mathrm{d}I \; \hat{\Phi}_T(I, \theta_0) \tag{29}$$

where the limits of integration are as defined previously. The operators $\hat{P}_\mathrm{r}^{\oplus}$, $\hat{P}_\mathrm{t}^{\oplus}$, $\hat{J}_\mathrm{t}^{\oplus}$ and $\hat{H}^{\oplus}$ are defined in the same way.

We can now formalise our description in terms of the following postulate.

Postulate 1. A TSCR under standard conditions subject to an applied current I flowing in at $\theta = 0$ and out at $\theta = \theta_0$ in the absence of any external magnetic fluxes possesses a continuous SSR so that its associated Hilbert space is the direct integral space $\mathcal{H}^{\oplus}$ and its observables are represented by decomposable self-adjoint operators in this space.

Postulate 2. The canonical momenta on each side of the ring are quantized as the operators $\hat{P}_\mathrm{l}^{\oplus}$ and $\hat{P}_\mathrm{r}^{\oplus}$. The through momentum is represented by the operator $\hat{P}_\mathrm{t}^{\oplus}$, the Hamiltonian by $\hat{H}^{\oplus}$, the through current by $\hat{J}_\mathrm{t}^{\oplus}$ and the total flux enclosed in the ring by $\hat{\Phi}_T^{\oplus}$.

Note that there is no destructive interference in the output current as a function of θ_0, in contrast to a double slit-type experiment for electrons. The behaviour of the present system is rather classical. Once external magnetic fluxes are included, however, quantum behaviour is exhibited in the form of the quantization of enclosed magnetic flux.

2.2. *Inclusion of external magnetic fields*

We now wish to incorporate the effects of a constant and uniform external magnetic field of magnitude B perpendicular to the plane of the ring (i.e. along the positive z-axis). In cylindrical coordinates (r, θ, z) the corresponding vector potential is $\boldsymbol{A} = (0, \frac{1}{2}Br, 0)$, which has magnitude $A(r) = \frac{1}{2}Br$. We simply replace the previous Hamiltonian by the new path-weighted Hamiltonian

$$H = \frac{1}{2m}\left[\frac{\theta_0}{2\pi}(p_\mathrm{l} - 2eA)^2 + \left(\frac{2\pi - \theta_0}{2\pi}\right)(p_\mathrm{r} - 2eA)^2\right] \tag{30}$$

where $A = A(a) = Ba/2$ is the magnitude of the vector potential at the ring. As before, we proceed by quantizing the left- and right-hand sides separately. The ring is still fed with a current I, but the presence of the external magnetic field changes the left and right currents I_l and I_r to new values I_l' and I_r'.

2.2.1. Quantization of the two sides. On the left, the Hilbert space is taken to be $\mathcal{H}_\mathrm{l} = L^2(\mathcal{S}_\mathrm{l})$ as defined previously. The canonical momentum is quantized as the selfadjoint operator $\hat{p}_{\varphi_\mathrm{l}'} = -(\mathrm{i}\hbar/a)(\mathrm{d}/\mathrm{d}\theta)$ on the domain

$$D_{\varphi_\mathrm{l}'} = \{\phi^1 : \phi^1 \in AC(\mathcal{S}_\mathrm{l});\; \phi^1(0_+) = \mathrm{e}^{\mathrm{i}\varphi_\mathrm{l}'}\phi^1(\theta_{0-});\; \hat{p}_{\varphi_\mathrm{l}'}\phi^1 \in L^2(\mathcal{S}_\mathrm{l})\}. \tag{31}$$

Left-hand current and left generated flux operators may be defined as [9]

$$\hat{J}_{\varphi_\mathrm{l}'} = \frac{e}{\pi a m}(\hat{p}_{\varphi_\mathrm{l}'} - 2eA\hat{\mathbb{I}}_\mathrm{l}) \qquad \hat{\Phi}_{\varphi_\mathrm{l}'} = L_\mathrm{l}\hat{J}_{\varphi_\mathrm{l}'} = \frac{\theta_0}{2\pi}\frac{\pi a}{e}(\hat{p}_{\varphi_\mathrm{l}'} - 2eA\hat{\mathbb{I}}_\mathrm{l}). \tag{32}$$

These operators share a common set of eigenfunctions

$$\psi^1_{\varphi_\mathrm{l}', k_\mathrm{l}'}(\theta) = \frac{1}{\sqrt{2\pi a}}\exp\left[\mathrm{i}\left(\frac{2\pi k_\mathrm{l}' - \varphi_\mathrm{l}'}{\theta_0}\right)\theta\right] \qquad k_\mathrm{l}' = 0, \pm 1 \ldots. \tag{33}$$

To facilitate a comparison with our earlier results in the absence of an external magnetic field we define $k_l' = k_l + \ell_l$ and $\varphi_l' = \varphi_l + \lambda_l$ where k_l and φ_l are the parameters determined by I and θ_0 (equation (10)) and are independent of A. Since $\varphi_l = \varphi_r = \varphi$, we may replace the subscript φ_l' by $\varphi + \lambda_l$ and rewrite the operators $\hat{p}_{\varphi_l'}$, $\hat{J}_{\varphi_l'}$, $\hat{\Phi}_{\varphi_l'}$ and $\hat{J}^{c}_{\varphi_l'}$ respectively as $\hat{p}_{\varphi,\lambda_l}$, $\hat{J}_{\varphi,\lambda_l}$, $\hat{\Phi}_{\varphi,\lambda_l}$ and $\hat{J}^{c}_{\varphi,\lambda_l}$. At present, λ_l is a real number in the interval $(-\pi, \pi]$. In due course we shall need to fix its value: it will turn out to be the phase due to the enclosed magnetic flux.

We can rewrite the eigenfunctions as

$$\psi^{l}_{I,\theta_0,\lambda_l,\ell_l}(\theta) = \frac{1}{\sqrt{2\pi a}} \exp\left[\mathrm{i}\left(\frac{2\pi\Phi_I}{\theta_0\Phi_0}\right)\theta\right]\exp\left[\mathrm{i}\left(\frac{2\pi\ell_l - \lambda_l}{\theta_0}\right)\theta\right] \qquad \ell_l = 0, \pm 1\ldots \tag{34}$$

with respective eigenvalues:

$$p_{I,\theta_0,\lambda_l,\ell_l} = \frac{\hbar}{a}\left(\frac{2\pi\Phi_I}{\theta_0\Phi_0} + \frac{2\pi\ell_l - \lambda_l}{\theta_0}\right)$$

$$j_{I,\theta_0,\lambda_l,\ell_l} = I_l + \frac{1}{L}\left[\Phi_0\left(\frac{2\pi\ell_l - \lambda_l}{\theta_0}\right) - \Phi_{ex}\right] \tag{35}$$

$$\Phi_{I,\theta_0,\lambda_l,\ell_l} = \Phi_I + \frac{\Phi_0}{2\pi}(2\pi\ell_l - \lambda_l) - \frac{\theta_0}{2\pi}\Phi_{ex} \tag{36}$$

where $\Phi_{ex} = 2\pi a A$ is the external flux applied to the ring.

The left-hand current eigenvalues (equation (35)) are composed of two parts: the applied current I_l, and an additional part which, together with the corresponding result for the right-hand path, may be interpreted as the screening current, i.e. the current circulating in the ring maintaining the difference between the enclosed and applied magnetic fluxes [16]. We therefore define a *circulating current* operator for the left-hand path:

$$\hat{J}^{c}_{\varphi,\lambda_l} = \hat{J}_{\varphi,\lambda_l} - I_l\hat{\mathbb{I}}_l \tag{37}$$

with eigenvalues

$$j^{c}_{I,\theta_0,\lambda_l,\ell_l} = \frac{1}{L}\left[\Phi_0\left(\frac{2\pi\ell_l - \lambda_l}{\theta_0}\right) - \Phi_{ex}\right]. \tag{38}$$

Let Φ_T be the total flux enclosed by the ring. The circulating current is responsible for maintaining the difference between the enclosed and externally applied fluxes, so should be equal to $(\Phi_T - \Phi_{ex})/L$. Equating this expression with the circulating current eigenvalues (equation (38)) leads to the following constraint on λ_l and ℓ_l:

$$(\Phi_T - \Phi_{ex}) = \Phi_0\left(\frac{2\pi\ell_l - \lambda_l}{\theta_0}\right) - \Phi_{ex} \Longrightarrow \frac{\theta_0\Phi_T}{2\pi\Phi_0} = \ell_l - \frac{\lambda_l}{2\pi}. \tag{39}$$

The quantity $\theta_0\Phi_T/2\pi\Phi_0$ is uniquely expressible as $N + \alpha$, where N is an integer and $-\frac{1}{2} \leqslant \alpha < \frac{1}{2}$. As ℓ_l is an integer and $-\frac{1}{2} \leqslant -(\lambda_l/2\pi) < \frac{1}{2}$, it follows that $k_l' = N$ and $\alpha = -\lambda_l/2\pi$. The values of ℓ_l and λ_l are thus completely determined by the enclosed flux Φ_T and the output lead position θ_0. We may express the momentum eigenfunctions as functions of the physical parameters I, θ_0 and Φ_T, rather than the formal parameters φ, λ_l, and ℓ_l:

$$\psi^{l}_{I,\theta_0,\Phi_T}(\theta) = \frac{1}{\sqrt{2\pi a}} \exp\mathrm{i}\left[\frac{2\pi\Phi_I}{\theta_0\Phi_0} + \frac{\Phi_T}{\Phi_0}\right]\theta. \tag{40}$$

This notation highlights the individual contributions to the wavefunction from the applied current and the enclosed magnetic flux.

Following an analogous procedure on the right, we define the operators $\hat{p}_{\varphi,\lambda_r}$, $\hat{J}_{\varphi,\lambda_r}$, $\hat{\Phi}_{\varphi,\lambda_r}$, and $\hat{J}^c_{\varphi,\lambda_r}$, representing respectively the canonical momentum, the right-hand current, the right generated flux and the circulating current.

As previously, both λ_r and the quantum number ℓ_r labelling the common eigenfunctions of these operators are fully determined by the enclosed flux and output lead position. In terms of the enclosed flux and applied current, the right-hand eigenfunctions are

$$\psi^r_{I,\theta_0,\Phi_T}(\theta) = \frac{1}{\sqrt{2\pi a}} \exp \mathrm{i} \left[\frac{2\pi \Phi_I}{(2\pi - \theta_0)\Phi_0} - \frac{\Phi_T}{\Phi_0} \right] (2\pi - \theta). \tag{41}$$

In requiring that the circulating current be the same on both sides, we have in effect imposed a constraint on the system, helping to link the left- and right-hand sides.

2.2.2. The whole system. We are now able to associate the entire ring with the direct sum of the individual Hilbert spaces: $\mathcal{H} = \mathcal{H}_l \oplus \mathcal{H}_r$. State vectors will be of the form $\psi^l_{I,\theta_0,\Phi_T}(\theta) \oplus \psi^r_{I,\theta_0,\Phi_T}(\theta)$. We have seen that states of the left- and right-hand sides are completely determined by the values of I, θ_0 and Φ_T. Written as functions on the circle $\mathcal{S}$, states of the entire ring will be of the form

$$\Psi_Q(\theta) = \Psi_{I,\theta_0,\Phi_T}(\theta) = \chi_l(\theta)\psi^l_{I,\theta_0,\Phi_T}(\theta) + \chi_r(\theta)\psi^r_{I,\theta_0,\Phi_T}(\theta) \tag{42}$$

where Q denotes any set of allowed parameter values (I, θ_0, Φ_T), or, equivalently, the corresponding set $(\varphi, \ell_l, \lambda_l, \lambda_r, \ell_r)$. In the previous case, where there was no external magnetic field, the left- and right-hand wavefunctions always matched up at their meeting points to give a continuous wavefunction over the entire ring. In this case, we use the constraint that the condensate wavefunction is single valued† around the ring [14, p 40] to ensure that the condensate wavefunction is continuous around the entire ring. This means that the left- and right-hand sides must match up at their meeting points, resulting in

$$\left(\frac{2\pi \Phi_I}{\theta_0 \Phi_0} + \frac{\Phi_T}{\Phi_0} \right) \theta_0 = \left(\frac{2\pi \Phi_I}{(2\pi - \theta_0)\Phi_0} - \frac{\Phi_T}{\Phi_0} \right) (2\pi - \theta_0) + 2n\pi \Rightarrow \Phi_T = n\Phi_0. \tag{43}$$

The only allowed states of the entire ring are those for which the total enclosed flux is quantized. As $\Phi_T = n\Phi_0$ always, we may relabel the functions $\Psi_{I,\theta_0,\Phi_T}(\theta)$, $\psi^l_{I,\theta_0,\Phi_T}(\theta)$ and $\psi^r_{I,\theta_0,\Phi_T}(\theta)$ as $\Psi_{I,\theta_0,n}(\theta)$, $\psi^l_{I,\theta_0,n}(\theta)$ and $\psi^r_{I,\theta_0,n}(\theta)$ respectively. We may also relabel the eigenvalue $p_{I,\theta_0,\lambda_r,\ell_r}$ as $p^l_{I,\theta_0,n}$, and $p_{I,\theta_0,\lambda_r,\ell_r}$ as $p^r_{I,\theta_0,n}$.

As previously, operators representing observables of only one side may be extended to the total Hilbert space $\mathcal{H}$. There is an additional complication here in that the external parameters (I, θ_0) do not uniquely determine the eigenfunctions and eigenvalues of the momentum operators. For a given (I, θ_0), the state vectors are of the form $\Psi_{I,\theta_0,n}(\theta) = \psi^l_{I,\theta_0,n}(\theta) \oplus \psi^r_{I,\theta_0,n}(\theta)$ for some n. The extended momentum operators must admit these

† Consider the example of the direct sum $L^2(\mathbb{R}) = L^2(-\infty, 0] \oplus L^2[0, \infty)$. Let $\phi(x) \in L^2(\mathbb{R})$ and $\phi = \phi_- \oplus \phi_+$ where $\phi_- \in L^2(-\infty, 0]$ and $\phi_+ \in L^2[0, \infty)$. Suppose ϕ_- is continuous in $(-\infty, 0]$ and ϕ_+ in $[0, \infty)$. By formally extending the functions ϕ_- and ϕ_+ to $\mathbb{R}$ with $\phi_-(x) = 0, x \in (0, \infty)$ and $\phi_+(x) = 0, x \in (-\infty, 0)$, we can write

$$\phi(x) = \chi_-(x)\phi_-(x) + \chi_+(x)\phi_+(x) \qquad x \in \mathbb{R}$$

where $\chi_-(x)$ and $\chi_+(x)$ are respectively the characteristic functions of the intervals $(-\infty, 0]$ and $(0, \infty)$. Alternatively we could have chosen χ_- and χ_+ respectively to be the characteristic functions of the intervals $(-\infty, 0)$ and $[0, \infty)$. This alternative choice will only produce the same value of $\phi(0)$ if $\phi_-(0) = \phi_+(0)$. So, the requirement of single valueness implies that $\phi_-(x)$ and $\phi_+(x)$ match up at the boundary point, which also results in the continuity of $\phi(x)$ in $\mathbb{R}$.

state vectors as eigenfunctions. This necessitates the introduction of new operators on $\mathcal{H}_{\mathrm{l}}$ and $\mathcal{H}_{\mathrm{r}}$ defined by

$$\begin{aligned}\hat{\wp}^{\mathrm{r}}_{I,\theta_0} &= \sum_n p^{\mathrm{l}}_{I,\theta_0,n}\hat{\wp}^{\mathrm{r}}_n \qquad \hat{\wp}^{\mathrm{r}}_n = |\psi^{\mathrm{r}}_{I,\theta_0,n}\rangle\langle\psi^{\mathrm{r}}_{I,\theta_0,n}| \text{ (projector on } \mathcal{H}_{\mathrm{r}}) \\ \hat{\wp}^{\mathrm{l}}_{I,\theta_0} &= \sum_n p^{\mathrm{r}}_{I,\theta_0,n}\hat{\wp}^{\mathrm{l}}_n \qquad \hat{\wp}^{\mathrm{l}}_n = |\psi^{\mathrm{l}}_{I,\theta_0,n}\rangle\langle\psi^{\mathrm{l}}_{I,\theta_0,n}| \text{ (projector on } \mathcal{H}_{\mathrm{l}}).\end{aligned} \tag{44}$$

To satisfy the two requirements concerning eigenvalues and eigenvectors of the operator extensions (requirements 2(a) and 2(b) in section 2.1.2), we make the following choices:

$$\begin{aligned}&\hat{P}^{\mathrm{l}}_{\varphi,\lambda_{\mathrm{l}}} = \hat{p}_{\varphi,\lambda_{\mathrm{l}}} \oplus \hat{\wp}^{\mathrm{r}}_{I,\theta_0} \qquad \hat{J}^{\mathrm{l}}_{\varphi,\lambda_{\mathrm{l}}} = \frac{e}{\pi a m}(\hat{P}_{\varphi,\lambda_{\mathrm{l}}} - 2eA\hat{\mathbb{I}}) \qquad \hat{\Phi}^{\mathrm{l}}_{\varphi,\lambda_{\mathrm{l}}} = L_{\mathrm{l}}\hat{J}^{\mathrm{l}}_{\varphi,\lambda_{\mathrm{l}}} \\ &\hat{P}^{\mathrm{r}}_{\varphi,\lambda_{\mathrm{r}}} = \hat{\wp}^{\mathrm{l}}_{I,\theta_0} \oplus \hat{p}_{\varphi,\lambda_{\mathrm{r}}} \qquad \hat{J}^{\mathrm{r}}_{\varphi,\lambda_{\mathrm{r}}} = \frac{e}{\pi a m}(\hat{P}_{\varphi,\lambda_{\mathrm{r}}} - 2eA\hat{\mathbb{I}}) \qquad \hat{\Phi}^{\mathrm{r}}_{\varphi,\lambda_{\mathrm{r}}} = L_{\mathrm{l}}\hat{J}^{\mathrm{r}}_{\varphi,\lambda_{\mathrm{r}}} \\ &\hat{J}^{\mathrm{cl}}_{\varphi,\lambda_{\mathrm{l}}} = \hat{J}^{\mathrm{l}}_{\varphi,\lambda_{\mathrm{l}}} - I_{\mathrm{l}}\hat{\mathbb{I}} \qquad \hat{J}^{\mathrm{cr}}_{\varphi,\lambda_{\mathrm{r}}} = \hat{J}^{\mathrm{r}}_{\varphi,\lambda_{\mathrm{r}}} + I_{\mathrm{r}}\hat{\mathbb{I}} \qquad \text{where } \hat{\mathbb{I}} = (\hat{\mathbb{I}}_{\mathrm{l}} \oplus \hat{\mathbb{I}}_{\mathrm{r}}).\end{aligned} \tag{45}$$

We have defined two operators representing the circulating current, one derived from the left and one from the right. Since they have the same eigenvalues, namely $(\Phi_T - \Phi_{\mathrm{ex}})/L$, we can define a circulating current operator for the ring as a whole:

$$\hat{J}^{\mathrm{c}}_{\varphi,\lambda_{\mathrm{l}},\lambda_{\mathrm{r}}} = \tfrac{1}{2}(\hat{J}^{\mathrm{cl}}_{\varphi,\lambda_{\mathrm{l}}} + \hat{J}^{\mathrm{cr}}_{\varphi,\lambda_{\mathrm{r}}}). \tag{46}$$

Likewise, we may define a circulating momentum operator in terms of either the left- or right-hand momentum:

$$\hat{P}^{\mathrm{cl}}_{\varphi,\lambda_{\mathrm{l}}} = \hat{P}^{\mathrm{l}}_{\varphi,\lambda_{\mathrm{l}}} - \frac{2e\Phi_I}{a\theta_0}\hat{\mathbb{I}} \qquad \hat{P}^{\mathrm{cr}}_{\varphi,\lambda_{\mathrm{r}}} = \hat{P}^{\mathrm{r}}_{\varphi,\lambda_{\mathrm{r}}} + \frac{2e\Phi_I}{a(2\pi-\theta_0)}\hat{\mathbb{I}}. \tag{47}$$

Since these two operators also have the same eigenvalues, we can introduce a circulating momentum operator for the whole ring:

$$\hat{P}^{\mathrm{c}}_{\varphi,\lambda_{\mathrm{l}},\lambda_{\mathrm{r}}} = \tfrac{1}{2}(\hat{P}^{\mathrm{cl}}_{\varphi,\lambda_{\mathrm{l}}} + \hat{P}^{\mathrm{cr}}_{\varphi,\lambda_{\mathrm{r}}}). \tag{48}$$

The Hamiltonian describing the system (equation (30)) is quantized as

$$\hat{H}_{\varphi,\lambda_{\mathrm{r}},\lambda_{\mathrm{r}}} = \frac{1}{2m}\left[\frac{\theta_0}{2\pi}(\hat{P}^{\mathrm{l}}_{\varphi,\lambda_{\mathrm{l}}} - 2eA\hat{\mathbb{I}})^2 + \left(\frac{2\pi-\theta_0}{2\pi}\right)\left(\hat{P}^{\mathrm{r}}_{\varphi,\lambda_{\mathrm{r}}} - 2eA\hat{\mathbb{I}}\right)^2\right]. \tag{49}$$

The total enclosed flux is equal to the sum of the fluxes generated on each side of the ring plus the externally applied flux Φ_{ex} and is represented by the operator

$$\hat{\Phi}_{T_{\varphi,\lambda_{\mathrm{l}},\lambda_{\mathrm{r}}}} = \hat{\Phi}^{\mathrm{l}}_{\varphi,\lambda_{\mathrm{l}}} + \hat{\Phi}^{\mathrm{r}}_{\varphi,\lambda_{\mathrm{r}}} + \Phi_{\mathrm{ex}}\hat{\mathbb{I}}. \tag{50}$$

As we might expect from our previous expression for the total enclosed flux, an equivalent definition may be given in terms of the circulating momentum:

$$\hat{\Phi}_{T_{\varphi,\lambda_{\mathrm{l}},\lambda_{\mathrm{r}}}} = \frac{\pi a}{e}\hat{P}^{\mathrm{c}}_{\varphi,\lambda_{\mathrm{l}},\lambda_{\mathrm{r}}}. \tag{51}$$

We may also define through momentum and through current operators as

$$\hat{P}^{\mathrm{t}}_{\varphi,\lambda_{\mathrm{l}},\lambda_{\mathrm{r}}} = \hat{P}_{\varphi,\lambda_{\mathrm{l}}} - \hat{P}_{\varphi,\lambda_{\mathrm{r}}} \qquad \hat{J}^{\mathrm{t}}_{\varphi,\lambda_{\mathrm{l}},\lambda_{\mathrm{r}}} = \hat{J}_{\varphi,\lambda_{\mathrm{l}}} - \hat{J}_{\varphi,\lambda_{\mathrm{r}}} = \frac{e}{\pi a m}\hat{P}^{\mathrm{t}}_{\varphi,\lambda_{\mathrm{l}},\lambda_{\mathrm{r}}}. \tag{52}$$

All these operators share a complete set of common eigenfunctions $\Psi_Q(\theta) = \Psi_{I,\theta_0,n}(\theta)$. Respective eigenvalues are:

$$\begin{aligned}
&p^{\rm l}_Q = \frac{\hbar}{a}\left(\frac{2\pi\Phi_I}{\theta_0\Phi_0} + n\right) \qquad p^{\rm r}_Q = -\frac{\hbar}{a}\left(\frac{2\pi\Phi_I}{(2\pi-\theta_0)\Phi_0} - n\right)\\
&p^{\rm c}_Q = p^{\rm cl}_Q = p^{\rm cr}_Q = \frac{\hbar n}{a} \qquad p^{\rm t}_Q = \frac{\pi a m}{e} I\\
&j^{\rm l}_Q = I_{\rm l} + \frac{1}{L}(n\Phi_0 - \Phi_{\rm ex}) \qquad j^{\rm r}_Q = -I_{\rm r} + \frac{1}{L}(n\Phi_0 - \Phi_{\rm ex})\\
&j^{\rm c}_Q = j^{\rm cl}_Q = j^{\rm cr}_Q = \frac{1}{L}(n\Phi_0 - \Phi_{\rm ex}) \qquad j^{\rm t}_Q = I_{\rm l} + I_{\rm r} = I \qquad \Phi_{T_Q} = n\Phi_0\\
&E_Q = \frac{1}{2L_{\rm l}}\left(\frac{\theta_0}{2\pi}(n\Phi_0 - \Phi_{\rm ex}) + \Phi_I\right)^2 + \frac{1}{2L_{\rm r}}\left(\frac{(2\pi-\theta_0)}{2\pi}(n\Phi_0 - \Phi_{\rm ex}) + \Phi_I\right)^2.
\end{aligned} \tag{53}$$

The requirement that the bulk superconductivity of the ring is not destroyed will place restrictions on the number of flux quanta that may be enclosed by the ring [9]. For given values of (I, θ_0), the system has a number of states available to it, but the total left- and right-hand currents must both remain below the bulk critical supercurrent $J_{\rm c}$:

$$\left|\frac{\theta_0}{2\pi} I + n\frac{\Phi_0}{L}\right| < J_c \qquad \text{and} \qquad \left|\frac{(2\pi-\theta_0)}{2\pi} I - n\frac{\Phi_0}{L}\right| < J_{\rm c}. \tag{54}$$

The number of enclosed flux quanta n must satisfy both these inequalities, which define integer upper and lower limits N^+_{I,θ_0} (positive) and N^-_{I,θ_0} (negative) on n. For given (I, θ_0), we postulate that the system is associated with the Hilbert space $\oplus_n\mathcal{H}(I, \theta_0, n)$, where $\mathcal{H}(I, \theta_0, n)$ is the one-dimensional subspace spanned by $\Psi_{I,\theta_0,n}(\theta)$ and the summation is over the range $n \in (N^-_{I,\theta_0}, N^+_{I,\theta_0})$.

2.2.3. Superselection rules. If the superposition principle were to apply here, the ring could be in a superposition $\Psi_{I,\theta_0,n} + \Psi_{I,\theta_0,m}$ $(n \neq m)$ of states of different enclosed flux quanta. In order to arrive at a correct description of a TSCR, which is always found to enclose a definite integer number of flux quanta, we need to introduce a SSR forbidding coherent superpositions of states of different n. The supersectors will be the one-dimensional subspaces $\mathcal{H}(I, \theta_0, n)$. For any fixed pair (I, θ_0), the system will possess a discrete SSR and be associated with the direct sum space $\oplus_n\mathcal{H}(I, \theta_0, n)$.

We may unify our description further by introducing continuous SSRs over the possible values of I and θ_0. We follow the same procedure as in the previous section to construct a direct integral with respect to the measures $\mathrm{d}I$ and $\mathrm{d}\theta_0$, where the integrals are over the permitted ranges of θ_0 and I:

$$\mathcal{H}^{\oplus} = \int^{\oplus} \mathrm{d}\theta_0 \int^{\oplus} \oplus_n\mathcal{H}(I, \theta_0)\,\mathrm{d}I. \tag{55}$$

For any state $\Psi_{I,\theta_0,n}(\theta)$, the values of φ, $\lambda_{\rm l}$ and $\lambda_{\rm r}$ are completely fixed and hence also the set of operators $\hat{P}^{\rm l}_{\varphi,\lambda_{\rm l}}$, $\hat{P}^{\rm r}_{\varphi,\lambda_{\rm r}}$, $\hat{P}^{\rm c}_{\varphi,\lambda_{\rm l},\lambda_{\rm r}}$, $\hat{P}^{\rm t}_{\varphi,\lambda_{\rm l},\lambda_{\rm r}}$, $\hat{J}^{\rm c}_{\varphi,\lambda_{\rm l},\lambda_{\rm r}}$, $\hat{J}^{\rm t}_{\varphi,\lambda_{\rm l},\lambda_{\rm r}}$, $\hat{H}_{\varphi,\lambda_{\rm l},\lambda_{\rm r}}$ and $\hat{\Phi}_{T_{\varphi,\lambda_{\rm l},\lambda_{\rm r}}}$. Let $\hat{P}^{\rm l}(I, \theta_0, n)$, $\hat{P}^{\rm r}(I, \theta_0, n)$, $\hat{P}^{\rm c}(I, \theta_0, n)$, $\hat{P}^{\rm t}(I, \theta_0, n)$, $\hat{J}^{\rm c}(I, \theta_0, n)$, $\hat{J}^{\rm t}(I, \theta_0, n)$, $\hat{H}(I, \theta_0, n)$, and $\hat{\Phi}_T(I, \theta_0, n)$ be their respective reductions to the subspace $\mathcal{H}(I, \theta_0, n)$. Then we can define direct integral operators acting on $\mathcal{H}^{\oplus}$ as follows

$$\hat{P}^{\oplus}_{\rm l} = \int^{\oplus} \mathrm{d}\theta_0 \int^{\oplus} \oplus_n \hat{P}^{\rm l}(I, \theta_0, n)\,\mathrm{d}I \qquad \hat{\Phi}^{\oplus}_T = \int^{\oplus} \mathrm{d}\theta_0 \int^{\oplus} \oplus_n \hat{\Phi}_T(I, \theta_0, n)\,\mathrm{d}I \tag{56}$$

and likewise for $\hat{P}^{\oplus}_{\rm r}$, $\hat{P}^{\oplus}_{\rm c}$, $\hat{P}^{\oplus}_{\rm t}$, $\hat{J}^{\oplus}_{\rm c}$, $\hat{J}^{\oplus}_{\rm t}$ and $\hat{H}^{\oplus}$.

We now make the following postulates.

Postulate 3. A TSCR under standard conditions subject to a current I applied at $\theta = 0$ and flowing out at θ_0 possesses continuous and discrete SSRs so that its associated Hilbert space is the direct integral space $\mathcal{H}^{\oplus}$ with all its observables represented by decomposable self-adjoint operators in this space.

Postulate 4. The left and right canonical momenta are quantized as the operators $\hat{P}_{\mathrm{l}}^{\oplus}$ and $\hat{P}_{\mathrm{r}}^{\oplus}$ respectively. The Hamiltonian is quantized as the operator $\hat{H}^{\oplus}$. The through momentum, through current, circulating momentum and circulating current are represented respectively by the operators $\hat{P}_{\mathrm{t}}^{\oplus}$, $\hat{J}_{\mathrm{t}}^{\oplus}$, $\hat{P}_{\mathrm{c}}^{\oplus}$ and $\hat{J}_{\mathrm{c}}^{\oplus}$, and the total enclosed flux by the operator $\hat{\Phi}_{T}^{\oplus}$.

The system has the following properties.

(1) Pure states, in the sense of Dirac delta function normalization in the direct integral space [23], correspond one-to-one to the eigenfunctions $\Psi_{I,\theta_0,n}(\theta)$, or, equivalently, to the one-dimensional subspaces $\mathcal{H}(I, \theta_0, n)$.

(2) As in the case of a uniform TSCR with no applied current [9], the position operator is not decomposable so does not represent an observable: the pseudoparticle representing the condensate is wholly delocalized around the ring, so the question as to which path the condensate took does not arise.

(3) For a given external environment $(I, \theta_0, \Phi_{\mathrm{ex}})$, the system has a number of states available to it, distinguished by n. The system's state is independent of Φ_{ex} but is partly determined by I and θ_0.

In contrast to the Aharonov–Bohm effect, here we *always* have constructive interference [24, p 6]. It follows from the assumption that the wavefunction around the ring is single valued and that the separate wavefunctions on the left and right have to be in phase at their meeting point. In this case, it is clear that the externally applied magnetic field plays no role in the phase matching at all. As can be seen from the pure-state wavefunctions, the relevant factors are the enclosed flux $n\Phi_0$ and the flux Φ_I, itself determined by the applied current I and the output lead position θ_0. These two fluxes are independent of each other: they each separately satisfy the phase matching condition. The enclosed flux achieves this by being quantized, the Φ_I component by the current splitting ratio. Hence, the system is able to remain superconducting for any combination of allowed values of (I, θ_0) and Φ_{ex}. The value of the output current is independent of both the applied flux and the output lead position: it exhibits no Aharonov–Bohm-type interference pattern, either as a function of Φ_{ex} or θ_0.

This system clearly possesses both classical and quantum characteristics. It does not entirely follow classical theory: quantum mechanics is required to obtain quantization of the enclosed flux.

3. dc SQUIDs: TSCRs containing two JJs

Having characterized the effect of an applied current on a TSCR in a static magnetic field, we now turn our attention to a dc SQUID: a TSCR containing two JJs which are parallel with respect to an applied dc supercurrent (figure 2). The maximum value of the dc supercurrent that can be passed through the ring is found to display an interference pattern as a function of the externally applied magnetic flux Φ_{ex} enclosed by the ring. For the sake of simplicity, we shall consider a symmetric device, with equal path lengths and identical JJs. The standard phenomonology (which is what we are trying to axiomatise) leads to equations for non-equal path lengths which cannot be solved analytically: computational methods show that

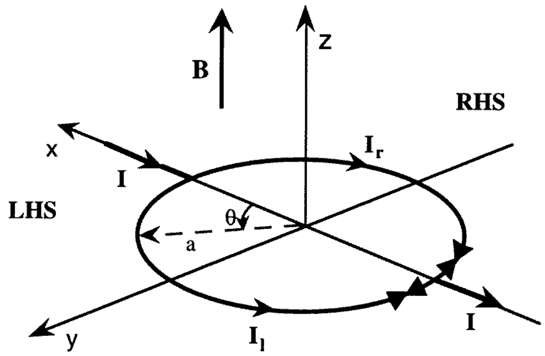

Figure 2. dc SQUID configuration.

the solutions are 'tilted' versions of the symmetric solutions [10, pp 375–82]. The exact position of the JJs along either of the paths does not affect the behaviour of the device as a whole. In a theoretical analysis of the device, it is most convenient to imagine that each JJ is located at one or other of the ends of the bulk intervals: we shall take them to be adjacent to the output lead.

Josephson's analysis of quantum tunnelling through thin insulating barriers (JJs) between superconductors [25–27] leads to the well known Josephson equation $j = \mathrm{i}_0 \sin\lambda$ for the dc supercurrent j through a JJ. The parameter λ is the phase difference between the wavefunctions of the superconductors on either side of the junction and i_0, known as the critical current of the junction, is the maximum (super)current that may be passed through it. The JJs in SQUIDs have critical currents satisfying $\mathrm{i}_0 < \Phi_0/2\pi L$, ensuring that their properties are unambiguously determined by Φ_{ex} [9, 14, 29]. Attempting to increase the current above the value i_0 leads to the appearance of a voltage across the JJ, the destruction of dc superconductivity, and various phenomena known collectively as ac Josephson effects [14].

In a dc SQUID the currents flowing through the left- and right-hand sides of the ring must obey the respective Josephson equations for the left- and right-hand junctions. There are two contributions to these currents: the applied current (which in equilibrium will equal the through, or output, current), and the circulating current j_{c} which maintains any difference between Φ_T, the total flux enclosed in the ring, and the externally applied flux Φ_{ex}. For a symmetric configuration, the applied current I will split equally between the left- and right-hand paths [16]. The total currents in the left- and right-hand paths are then $j_{\mathrm{l}} = I/2 + j_{\mathrm{c}}$ and $j_{\mathrm{r}} = I/2 - j_{\mathrm{c}}$ respectively. The requirement that the wavefunction be single valued around the entire ring implies that the left and right wavefunctions must match up at their meeting points. The total phases picked up along each path must be equal, modulo $2n\pi$ [11, 28]. These constraints lead to the standard equations characterizing the behaviour of a dc SQUID.

(1) The Josephson equation for the applied supercurrent passing through the ring (which by definition is positive):

$$I = \mathrm{i}_0 \sin\lambda_{\mathrm{l}} + \mathrm{i}_0 \sin\lambda_{\mathrm{r}} = 2\mathrm{i}_0 \sin\lambda_0 \cos\left(\frac{\pi\Phi_T}{\Phi_0}\right) \tag{57}$$

where λ_{l} and λ_{r} are respectively the phase drops across the left- and right-hand JJs, $\lambda_0 = (\lambda_{\mathrm{l}} + \lambda_{\mathrm{r}})/2$, and λ_{l}, λ_{l} and Φ_T are related by [16, 28]:

$$\lambda_{\mathrm{l}} = \lambda_0 - \pi\Phi_T/\Phi_0 + n\pi \qquad \lambda_{\mathrm{r}} = \lambda_0 + \pi\Phi_T/\Phi_0 - n\pi \qquad \text{for some integer } n. \tag{58}$$

(2) The related equation for the maximum through supercurrent I_c (also known as the critical current of the device), which occurs when $|\sin\lambda_0| = 1$ and $\Phi_T = \Phi_{ex}$:

$$I_c = 2i_0 \left|\cos\left(\frac{\pi\Phi_{ex}}{\Phi_0}\right)\right|. \tag{59}$$

The critical current of the device is thus a Φ_0-periodic function of the applied flux, with minima of zero when $\Phi_{ex} = (n \pm \frac{1}{2})\Phi_0$ and maxima when $\Phi_{ex} = n\Phi_0$. It is this result which enables dc SQUIDs to operate as sensitive magnetometers. Small changes in a background magnetic field may be detected via measurements of the maximum current that can be passed through the ring without a potential difference appearing across it [10].

(3) The *flux–current* relation: $\Phi_T = \Phi_{ex} + Lj_c$ where L is the inductance of the whole ring [11].

(4) The *flux quantization* condition (although the enclosed flux is actually not discretized at all, having a continuous range of values): $\Phi_T = [n - (\lambda_l - \lambda_r)/2\pi]\Phi_0$.

We shall take these equations as defining the system and shall now show that it may be modelled axiomatically within the mathematical framework which we have developed.

3.1. Modelling the system

We again start with a classical Hamiltonian that is an equally weighted sum of the energies of the two possible classical paths.

$$H = \frac{1}{4m}[(p_l - 2eA)^2 + (p_r - 2eA)^2] - \frac{i_0\hbar}{2e}\left[\cos\left(\frac{\pi a}{\hbar}p_l\right) + \cos\left(\frac{\pi a}{\hbar}p_r\right)\right] \tag{60}$$

where the energies of the two JJs are characterized by the two momentum-dependent cosine terms [9, 29–33]. In the terminology of Spiller *et al* [36], a Hamiltonian of this form, in which the inductive energy plays the standard kinetic energy role, corresponds to the 'magnetic' picture rather than the 'electric' picture. As Spiller *et al* [36] showed, this is a sensible choice when dealing with through currents (continuous charges) and quantized fluxes, that although it is less obvious how junction capacitance (which we neglect in our treatment) might be incorporated into this Hamiltonian†. The electric picture, in which the capacitive energy acts as the kinetic energy term, thus facilitating treatment of junction capacitances, is more appropriate for describing charge (rather than flux) quantized systems, such as an isolated junction.

We assume that the JJs subtend negligibly small angles so that the bulk superconducting sections on the left and right may be taken as occupying the intervals $\theta \in [0, \pi)$ and $\theta \in (\pi, 2\pi]$ respectively. We proceed with quantization as in the previous sections, taking the Hilbert space $\mathcal{H}_l$ to be the space $L^2(\mathcal{S}_l)$ of square-integrable functions on the interval $\mathcal{S}_l = \{\theta \in [0, \pi)\}$ with respect to the measure $a\,d\theta$. The canonical momentum is represented by the operator $\hat{p}_{\varphi_l} = -(i\hbar/a)(d/d\theta)$ which is self-adjoint on the domain

$$D_{\varphi_l} = \{\phi^l : \phi^l \in AC(\mathcal{S}_l) : \phi^l(0) = e^{i\varphi_l}\phi^l(\pi_-);\ \hat{p}_{\varphi_l}\phi^l \in L^2(\mathcal{S}_l)\} \tag{61}$$

where $AC(\mathcal{S}_l)$ denotes the set of absolutely continuous functions $\phi^l(\theta)$ on the interval $\mathcal{S}_l$ and φ_l is a real number in the interval $(-\pi, \pi]$.

† There are different ways to introduce an electric flux operator $\hat{Q}$ [37]. For example, a common practice would be to use the commutation relation between the electric flux and the magnetic flux operators: $[\hat{\Phi}, \hat{Q}] = i\hbar$. In our formulation $\hat{\Phi}$ is linear in $\hat{p}_{\varphi_\ell}$; clearly an appropriate operator linear in φ_ℓ will satisfy the above commutation relation. The technicalities involved will be discussed elsewhere.

The total left-hand current and the magnetic flux generated by this current are represented by

$$\hat{J}_{\varphi_l} = \frac{e}{\pi a m}(\hat{p}_{\varphi_l} - 2eA\hat{\mathbb{I}}_l) \qquad \hat{\Phi}_{\varphi_l} = \frac{L}{2}\hat{J}_{\varphi_l} = \frac{\pi a}{2e}(\hat{p}_{\varphi_l} - 2eA\hat{\mathbb{I}}_l) \tag{62}$$

where we have again assumed that the inductance of the ring is uniform and hence that the inductance of the left-hand path is $L_l = L/2$. These operators share common eigenfunctions

$$\psi^l_{\varphi_l,k}(\theta) = \frac{1}{\sqrt{2\pi a}} \exp \mathrm{i}\left[\frac{2\pi k - \varphi_l}{\pi}\right]\theta \qquad k = 0, \pm 1 \pm 2, \ldots \tag{63}$$

with eigenvalues:

$$\begin{aligned} p_{\varphi_l,k} &= \frac{\hbar}{\pi a}(2\pi k - \varphi_l) \qquad j_{\varphi_l,k} = \frac{1}{L}\left[\frac{(2\pi k - \varphi_l)}{\pi}\Phi_0 - \Phi_{ex}\right] \\ \Phi_{\varphi_l,k} &= \frac{(2\pi k - \varphi_l)}{2\pi}\Phi_0 - \frac{\Phi_{ex}}{2}. \end{aligned} \tag{64}$$

The Hamiltonian on the left-hand side is taken to be

$$H_l = \frac{1}{4m}(p_l - 2eA)^2 - \frac{\mathrm{i}_0\hbar}{2e}\cos\left(\frac{\pi a}{\hbar}p_l\right) \tag{65}$$

and is quantized as

$$\hat{H}_{\varphi_l} = \frac{1}{4m}(\hat{p}_{\varphi_l} - 2eA\hat{\mathbb{I}}_l)^2 - \frac{i_0\hbar}{2e}\cos\left(\frac{\pi a}{\hbar}\hat{p}_{\varphi_l}\right) \tag{66}$$

with eigenvalues

$$E_{\varphi_l,k} = \frac{1}{4L}\left[\Phi_0\left(\frac{2\pi k - \varphi_l}{\pi}\right) - \Phi_{ex}\right]^2 - \frac{\mathrm{i}_0\hbar}{2e}\cos\varphi_l. \tag{67}$$

On the right-hand side we take $\mathcal{H}_r$ to be the space $L^2(\mathcal{S}_r)$ of square-integrable functions on the interval $\mathcal{S}_r = \{\theta \in (\pi, 2\pi]\}$. We define the selfadjoint operator $\hat{p}_{\varphi_r} = -(\mathrm{i}\hbar/a))(\mathrm{d}/\mathrm{d}\theta)$ on the domain

$$D_{\varphi_r} = \{\phi^r : \phi^r \in AC(\mathcal{S}_r) : \phi^r(\pi_+) = \mathrm{e}^{-\mathrm{i}\varphi_r}\phi^r(2\pi);\ \hat{p}_{\varphi_r}\phi^r \in L^2(\mathcal{S}_r)\} \tag{68}$$

where $AC(\mathcal{S}_r)$ denotes the set of absolutely continuous functions $\phi^r(\theta)$ on the interval $\mathcal{S}_r$, and $\varphi_r \in (-\pi, \pi]$. We assume that the particle's canonical momentum is represented in $\mathcal{H}_r$ by the operator $\hat{p}_{\varphi_r}$ and we define the total right-hand current and right generated flux operators as

$$\hat{J}_{\varphi_r} = \frac{e}{\pi a m}(\hat{p}_{\varphi_r} - 2eA\hat{\mathbb{I}}_r) \qquad \hat{\Phi}_{\varphi_r} = \frac{\pi a}{2e}(\hat{p}_{\varphi_r} - 2eA\hat{\mathbb{I}}_r). \tag{69}$$

These operators share common eigenfunctions

$$\psi^r_{\varphi_r,k'}(\theta) = \frac{1}{\sqrt{2\pi a}} \exp \mathrm{i}\left[\frac{2\pi k' - \varphi_r}{\pi}\right](2\pi - \theta) \qquad k' = 0, \pm 1, \pm 2, \ldots \tag{70}$$

with respective eigenvalues:

$$\begin{aligned} p_{\varphi_r,k'} &= -\frac{\hbar}{\pi a}(2\pi k' - \varphi_r) \qquad j_{\varphi_r,k'} = -\frac{1}{L}\left[\left(\frac{2\pi k' - \varphi_r}{\pi}\right)\Phi_0 + \Phi_{ex}\right] \\ \Phi_{\varphi_r,k'} &= -\frac{1}{2}\left[\left(\frac{2\pi k' - \varphi_r}{\pi}\right)\Phi_0 + \Phi_{ex}\right]. \end{aligned} \tag{71}$$

Taking the Hamiltonian on the right-hand side to be

$$H_{\rm r} = \frac{1}{4m}(p_{\rm r} - 2eA)^2 - \frac{{\rm i}_0\overline{h}}{2e}\cos\left(\frac{\pi a}{\overline{h}}p_{\rm r}\right) \tag{72}$$

we quantize it as

$$\hat{H}_{\varphi_{\rm r}} = \frac{1}{4m}(\hat{p}_{\varphi_{\rm r}} - 2eA\hat{\mathbb{I}}_{\rm r})^2 4m - \frac{{\rm i}_0\overline{h}}{2e}\cos\left(\frac{\pi a}{\overline{h}}\hat{p}_{\varphi_{\rm r}}\right) \tag{73}$$

with eigenvalues

$$E_{\varphi_{\rm r}} = \frac{1}{4L}\left[\Phi_0\left(\frac{2\pi k' - \varphi_{\rm r}}{\pi}\right) - \Phi_{\rm ex}\right]^2 - \frac{{\rm i}_0\overline{h}}{2e}\cos\varphi_{\rm r}. \tag{74}$$

This system is not yet well defined, because we have not fixed the values of $\varphi_{\rm l}$ and $\varphi_{\rm r}$, nor do we have the Josephson equations. One way to solve these problems is by an energy minimization process [9, 34, 35]. If we minimize the eigenvalues $E_{\varphi_{\rm l}}$ with respect to $\varphi_{\rm l}$ we obtain the Josephson equation for the left-hand side, and minimization of $E_{\varphi_{\rm r}}$ with respect to $\varphi_{\rm r}$ yields the Josephson equation for the right-hand side:

$$j_Q^{\rm l} = {\rm i}_0 \sin\varphi_{\rm l} \qquad - j_Q^{\rm r} = {\rm i}_0\sin\varphi_{\rm r}. \tag{75}$$

We now see that $\varphi_{\rm l}$ and $\varphi_{\rm r}$ may be identified with the traditional phase differences $\lambda_{\rm l}$ and $\lambda_{\rm r}$ across the JJs in the left- and right-hand paths, and the usual Josephson equation applies to both paths separately. Our assumption that the applied current flowing in at $\theta = 0$ is positive effectively restricts the values of the phases $\varphi_{\rm l}$, $\varphi_{\rm r}$ to the range $(0, \pi]$, so that $j_Q^{\rm l}$ is positive and $j_Q^{\rm r}$ is negative.

3.1.1. The whole ring. The whole ring is associated with the direct sum space $\mathcal{H} = \mathcal{H}_{\rm l} \oplus \mathcal{H}_{\rm r}$. State vectors are assumed to be of the form $\Psi_{\varphi_{\rm l},\varphi_{\rm r},k,k'} = \psi^{\rm l}_{\varphi_{\rm l},k} \oplus \psi^{\rm r}_{\varphi_{\rm r},k'}$. Operators defined on one or the other of the subspaces may be formally extended to the entire space subject to the two requirements stated previously. Let $\hat{P}^{\rm l}_{\varphi_{\rm l}}$ and $\hat{P}^{\rm r}_{\varphi_{\rm r}}$ be the extensions of the two momentum operators, which we construct in the following manner. For the sake of simplicity, let us confine ourselves to a small applied flux, $\Phi_{\rm ex} \in (-\Phi_0/2, \Phi_0/2)$. When $\Phi_{\rm ex} = 0$, symmetry and the flux quantization condition imply $j_Q^{\rm c} = 0$ and $j_{\varphi_{\rm l},k} = j_{\varphi_{\rm r},k'}$

$$\Rightarrow \Phi_T = 0 \Rightarrow \varphi_{\rm l} = \varphi_{\rm r} \qquad \text{and} \qquad k = k' \Rightarrow \tfrac{1}{2}I = {\rm i}_0\sin\varphi_{\rm l} = {\rm i}_0\sin\varphi_{\rm r}. \tag{76}$$

As $|\Phi_{\rm ex}|$ increases from zero, $\varphi_{\rm l} \neq \varphi_{\rm r}$, leading to a non-zero enclosed flux Φ_T. The circulating current opposes $\Phi_{\rm ex}$, so that $|\Phi_T| < |\Phi_{\rm ex}|$, and $k = k'$ is unchanged. That is, under these conditions, a given external environment $(I, \Phi_{\rm ex})$, fully determines $\varphi_{\rm l}$ and $\varphi_{\rm r}$ and hence the states, operators and eigenvalues for both sides of the ring. The state vector is

$$\Psi_{I,\Phi_{\rm ex}} = \Psi_{\varphi_{\rm l},\varphi_{\rm r},k,k} = \psi^{\rm l}_{\varphi_{\rm l},k} \oplus \psi^{\rm r}_{\varphi_{\rm r},k}. \tag{77}$$

This makes it easy to construct the extensions of the momentum operators. We have

$$\hat{P}^{\rm l}_{\varphi_{\rm l}} = \hat{p}_{\varphi_{\rm l}} \oplus C^{\rm l}_{I,\Phi_{\rm ex}}\hat{\mathbb{I}}_{\rm r} \qquad \hat{P}^{\rm r}_{\varphi_{\rm r}} = C^{\rm r}_{I,\Phi_{\rm ex}}\hat{\mathbb{I}}_{\rm l} \oplus \hat{p}_{\varphi_{\rm r}} \tag{78}$$

where the constants $C^{\rm l}_{I,\Phi_{\rm ex}}, C^{\rm r}_{I,\Phi_{\rm ex}}$ are determined by the external parameters $(I, \Phi_{\rm ex})$ and are respectively the eigenvalues of $\hat{p}_{\varphi_{\rm l}}$ and $\hat{p}_{\varphi_{\rm r}}$. Other operator extensions are given by

$$\begin{aligned} \hat{J}^{\rm l}_{\varphi_{\rm l}} &= \frac{e}{\pi a m}(\hat{P}^{\rm l}_{\varphi_{\rm l}} - 2eA\hat{\mathbb{I}}) \qquad \hat{\Phi}^{\rm l}_{\varphi_{\rm l}} = \tfrac{1}{2}L\hat{J}^{\rm l}_{\varphi_{\rm l}} \\ \hat{J}^{\rm r}_{\varphi_{\rm r}} &= \frac{e}{\pi a m}(\hat{P}^{\rm r}_{\varphi_{\rm r}} - 2eA\hat{\mathbb{I}} \qquad \hat{\Phi}^{\rm r}_{\varphi_{\rm r}} = \tfrac{1}{2}L\hat{J}^{\rm r}_{\varphi_{\rm r}}. \end{aligned} \tag{79}$$

The Hamiltonian for the entire system is quantized as

$$\hat{H}_{\varphi_l,\varphi_r} = \frac{1}{4m}[(\hat{P}^l_{\varphi_l} - 2eA\hat{\mathbb{I}})^2 + (\hat{P}^r_{\varphi_r} - 2eA\hat{\mathbb{I}})^2] - \frac{i_0\hbar}{2e}\left[\cos\left(\frac{\pi a}{\hbar}\hat{P}^l_{\varphi_l}\right) + \cos\left(\frac{\pi a}{\hbar}\hat{P}^r_{\varphi_r}\right)\right]. \quad (80)$$

The through current will be the sum of the currents flowing through the left- and right-hand paths and is represented by the operator $\hat{J}^t_{\varphi_l,\varphi_r} = \hat{J}^l_{\varphi_l} - \hat{J}^r_{\varphi_r}$, where the negative sign is present because of the direction in which the right-hand current was defined (around, rather than through, the ring). We can also define the through current in terms of the through momentum operator $\hat{P}^t_{\varphi_l,\varphi_r} = \hat{P}^l_{\varphi_l} - \hat{P}^r_{\varphi_r}$

$$\hat{J}^t_{\varphi_l,\varphi_r} = \frac{e}{\pi am}\hat{P}^t_{\varphi_l,\varphi_r}. \quad (81)$$

Likewise, the circulating current, represented by the operator $\hat{J}^c_{\varphi_l,\varphi_r}$, can be defined in terms of either the left and right current operators or the circulating momentum operator $\hat{P}^c_{\varphi_l,\varphi_r} = (\hat{P}^l_{\varphi_l} + \hat{P}^r_{\varphi_r})/2$:

$$\hat{J}^c_{\varphi_l,\varphi_r} = \tfrac{1}{2}(\hat{J}^l_{\varphi_l} + \hat{J}^r_{\varphi_r}) \qquad \text{or} \qquad \hat{J}^c_{\varphi_l,\varphi_r} = \frac{e}{\pi am}(\hat{P}^c_{\varphi_l,\varphi_r} - 2eA\hat{\mathbb{I}}). \quad (82)$$

For both the through and circulating currents, the two alternative definitions yield the same sets of eigenvectors and eigenvalues. There are also various equivalent definitions of the total enclosed flux operator $\hat{\Phi}_{T_{\varphi_l,\varphi_r}}$:

$$\hat{\Phi}_{T_{\varphi_l,\varphi_r}} = \frac{L}{2}(\hat{J}^l_{\varphi_l} + \hat{J}^r_{\varphi_r}) + \Phi_{ex} = L\hat{J}^c_{\varphi_l,\varphi_r} + \Phi_{ex} = \frac{\pi a}{2e}(\hat{P}^l_{\varphi_l} + \hat{P}^r_{\varphi_r}) = \frac{\pi a}{e}\hat{P}^c_{\varphi_l,\varphi_r}. \quad (83)$$

All these operators share a common complete set of normalized eigenfunctions $\Psi_{\varphi_l,\varphi_r,k,k'}(\theta)$, which, written as functions on the entire ring, are of the form

$$\Psi_{\varphi_l,\varphi_r,k,k'}(\theta) = \frac{1}{\sqrt{2\pi a}}\left[\chi_l(\theta)\exp i\left(\frac{2\pi k - \varphi_l}{\pi}\right)\theta + \chi_r(\theta)\exp i\left(\frac{2\pi k' - \varphi_r}{\pi}\right)(2\pi - \theta)\right]. \quad (84)$$

Respective eigenvalues, where Q denotes any set of allowed values $(\varphi_l, \varphi_r, k, k')$, are:

$$\begin{aligned}
&p^l_Q = \frac{\hbar}{\pi a}(2\pi k - \varphi_l) \qquad j^l_Q = \frac{1}{L}\left(\frac{2\pi k - \varphi_l}{\pi}\Phi_0 - \Phi_{ex}\right)\\
&p^r_Q = -\frac{\hbar}{\pi a}(2\pi k' - \varphi_r) \qquad j^r_Q = -\frac{1}{L}\left(\frac{2\pi k' - \varphi_r}{\pi}\Phi_0 + \Phi_{ex}\right)\\
&p^t_Q = \frac{\hbar}{\pi a}(4\pi(k + k') - (\varphi_l + \varphi_r)) \qquad j^t_Q = j^l_Q - j^r_Q\\
&p^c_Q = -\frac{\hbar}{a}[2\pi(k - k') - (\varphi_l - \varphi_r)] \qquad j^c_Q = \frac{1}{2}(j^l_Q + j^r_Q)\\
&E_Q = \frac{1}{4L}\left[\left(\frac{2\pi k - \varphi_l}{\pi}\Phi_0 - \Phi_{ex}\right)^2 + \left(\frac{2\pi k' - \varphi_r}{\pi}\Phi_0 + \Phi_{ex}\right)^2\right] - \frac{i_0\hbar}{2e}(\cos\varphi_l + \cos\varphi_r)\\
&\Phi_{T_Q} = \left[(k - k') - \left(\frac{\varphi_l - \varphi_r}{2\pi}\right)\right]\Phi_0.
\end{aligned} \quad (85)$$

We can now identify φ_l, φ_r and $k - k'$ respectively with λ_l, λ_r and n to obtain the traditional equations characterizing a dc SQUID. Finally, by writing φ_l and φ_r as

$$\varphi_l = \varphi_0 - \frac{\pi\Phi_{T_Q}}{\Phi_0} \qquad \varphi_r = \varphi_0 + \frac{\pi\Phi_{T_Q}}{\Phi_0} \quad (86)$$

we obtain

$$j_Q^{\mathrm{t}} = 2\mathrm{i}_0 \sin\varphi_0 \cos\left(\frac{\pi\Phi_{T_Q}}{\Phi_0}\right) \qquad j_Q^{\mathrm{c}} = -\,\mathrm{i}_0 \sin\frac{\pi\Phi_{T_Q}}{\Phi_0}\cos\varphi_0. \tag{87}$$

In equilibrium, the input current I will be equal to the output (through) current eigenvalues j_Q^{t}. The maximum current I_{c} that can be passed through the ring as a supercurrent corresponds to $\varphi_0 = \pi/2$:

$$I_{\mathrm{c}} = 2\mathrm{i}_0 \left|\cos\left(\frac{\pi\Phi_{T_Q}}{\Phi_0}\right)\right|. \tag{88}$$

When the value of the through current is at its maximum value I_{c}, the circulating current j_Q^{c} is zero so the enclosed flux will be equal to the applied flux. The critical current I_{c} can thus be expressed as a function of the externally applied flux Φ_{ex}, as previously:

$$I_{\mathrm{c}}(\Phi_{\mathrm{ex}}) = 2\mathrm{i}_0 \left|\cos\left(\frac{\pi\Phi_{\mathrm{ex}}}{\Phi_0}\right)\right|. \tag{89}$$

3.2. Superselection rules

For a given external environment (I, Φ_{ex}), the state of the system, the operators $\hat{P}_{\varphi_{\mathrm{l}}}^{\mathrm{l}}$ and $\hat{P}_{\varphi_{\mathrm{r}}}^{\mathrm{r}}$ and their specific eigenvalues are all fully determined. We have a one-state quantum system and there is no possibility of the system existing in a state which is a superposition of different $\Psi_{I,\Phi_{\mathrm{ex}}}$. We can formalise this situation in terms of SSRs.

For any particular value of $\Phi_{\mathrm{ex}} \in (-\Phi_0/2, \Phi_0/2)$, the applied supercurrent I is limited by the critical current $I_{\mathrm{c}}(\Phi_{\mathrm{ex}})$ (equation (89)) to lie within the range $I \in (0, I_{\mathrm{c}}(\Phi_{\mathrm{ex}}))$. Let $\mathcal{H}(I, \Phi_{\mathrm{ex}})$ be the one-dimensional subspace spanned by the eigenfunction $\Psi_{I,\Phi_{\mathrm{ex}}}$. We can construct a double direct integral of $\mathcal{H}(I, \Phi_{\mathrm{ex}})$ with respect to the measure $\mathrm{d}I\,\mathrm{d}\Phi_{\mathrm{ex}}$ [22]:

$$\mathcal{H}^{\oplus} = \int^{\oplus} \mathrm{d}\Phi_{\mathrm{ex}} \int^{\oplus} \mathcal{H}(I, \Phi_{\mathrm{ex}})\,\mathrm{d}I \tag{90}$$

where the integrals hereafter are over the ranges $\Phi_{\mathrm{ex}} \in (-\Phi_0/2, \Phi_0/2)$ and $I \in (0, I_{\mathrm{c}}(\Phi_{\mathrm{ex}}))$.

For any given external environment (I, Φ_{ex}), the choice of operators $\hat{P}_{\varphi_{\mathrm{l}}}^{\mathrm{l}}$, $\hat{P}_{\varphi_{\mathrm{r}}}^{\mathrm{r}}$, $\hat{P}_{\varphi_{\mathrm{l}},\varphi_{\mathrm{r}}}^{\mathrm{t}}$, $\hat{P}_{\varphi_{\mathrm{l}},\varphi_{\mathrm{r}}}^{\mathrm{c}}$, $\hat{J}_{\varphi_{\mathrm{l}},\varphi_{\mathrm{r}}}^{\mathrm{t}}$, $\hat{J}_{\varphi_{\mathrm{l}},\varphi_{\mathrm{r}}}^{\mathrm{c}}$, $\hat{H}_{\varphi_{\mathrm{l}},\varphi_{\mathrm{r}}}$ and $\hat{\Phi}_{T_{\varphi_{\mathrm{l}},\varphi_{\mathrm{r}}}}$ is fixed, as are their eigenvalues. Let the operators $\hat{P}^{\mathrm{l}}(I, \Phi_{\mathrm{ex}})$, $\hat{P}^{\mathrm{r}}(I, \Phi_{\mathrm{ex}})$, $\hat{P}^{\mathrm{t}}(I, \Phi_{\mathrm{ex}})$, $\hat{P}^{\mathrm{c}}(I, \Phi_{\mathrm{ex}})$, $\hat{J}^{\mathrm{t}}(I, \Phi_{\mathrm{ex}})$, $\hat{J}^{\mathrm{c}}(I, \Phi_{\mathrm{ex}})$, $\hat{H}(I, \Phi_{\mathrm{ex}})$ and $\hat{\Phi}_T(I, \Phi_{\mathrm{ex}})$ be their respective reductions to the subspace $\mathcal{H}(I, \Phi_{\mathrm{ex}})$. Then we may construct direct integral operators acting on $\mathcal{H}^{\oplus}$ as follows

$$\hat{P}_{\mathrm{l}}^{\oplus} = \int^{\oplus} \mathrm{d}\Phi_{\mathrm{ex}} \int^{\oplus} \hat{P}^{\mathrm{l}}(I, \Phi_{\mathrm{ex}})\,\mathrm{d}I \qquad \hat{\Phi}_T^{\oplus} = \int^{\oplus} \mathrm{d}\Phi_{\mathrm{ex}} \int^{\oplus} \hat{\Phi}_T(I, \Phi_{\mathrm{ex}})\,\mathrm{d}I \tag{91}$$

and likewise $\hat{P}_{\mathrm{r}}^{\oplus}$, $\hat{P}_{\mathrm{t}}^{\oplus}$, $\hat{P}_{\mathrm{c}}^{\oplus}$, $\hat{J}_{\mathrm{t}}^{\oplus}$, $\hat{J}_{\mathrm{c}}^{\oplus}$ and $\hat{H}^{\oplus}$.

We can now formalise the system in terms of the following postulate.

Postulate 5. A TSCR containing two JJs in a dc SQUID configuration with $\Phi_{\mathrm{ex}} \in (-\Phi_0/2, \Phi_0/2)$ possesses continuous SSR so that its associated Hilbert space is the direct integral space $\mathcal{H}^{\oplus}$ with all its observables represented by decomposable (i.e. direct integral) self-adjoint operators in this space.

Postulate 6. The canonical momenta on the left- and right-hand sides of the dc SQUID are quantized as the operators $\hat{P}_{\mathrm{l}}^{\oplus}$ and $\hat{P}_{\mathrm{r}}^{\oplus}$ respectively, and the Hamiltonian as $\hat{H}^{\oplus}$. The through momentum, through current, circulating momentum and circulating current are represented respectively by the operators $\hat{P}_{\mathrm{t}}^{\oplus}$, $\hat{J}_{\mathrm{t}}^{\oplus}$, $\hat{P}_{\mathrm{c}}^{\oplus}$ and $\hat{J}_{\mathrm{c}}^{\oplus}$, and the enclosed flux by the operator $\hat{\Phi}_T^{\oplus}$.

Our model possesses the following properties.

(1) Pure states, in the sense of normalized Dirac delta functions in the direct integral space $\mathcal{H}^{\oplus}$ [23], correspond one-to-one to the one-dimensional subspaces $\mathcal{H}(I, \Phi_{\mathrm{ex}})$; that is, to eigenfunctions $\Psi_{I,\Phi_{\mathrm{ex}}}(\theta)$. Linear combinations represent mixed states.

(2) A quantum system comprises the quantum object plus its static environment and usually has a number of states available to it [9, p 5]. But here we have a one-state system for each external environment (I, Φ_{ex}).

(3) The standard equations characterizing the behaviour of the device are obeyed.

(a) The Josephson equation for the through supercurrent:

(i) in operator form $\hat{J}_{\mathrm{t}}^{\oplus} = 2\mathrm{i}_0 \sin\lambda_0 \cos(\pi\hat{\Phi}_T^{\oplus}/\Phi_0)$;

(ii) in eigenvalue form $j_Q^{\mathrm{t}} = 2\mathrm{i}_0 \sin\lambda_0 \cos(\pi\Phi_{T_Q}/\Phi_0)$.

(b) The critical current equation: $I_{\mathrm{c}}(\Phi_{\mathrm{ex}}) = 2\mathrm{i}_0|\cos(\frac{\pi\Phi_{\mathrm{ex}}}{\Phi_0})|$.

(c) The flux-current relation:

(i) in operator form $\hat{\Phi}_T^{\oplus} = \int^{\oplus} \mathrm{d}\Phi_{\mathrm{ex}} \int^{\oplus} \mathrm{d}I\, [\Phi_{\mathrm{ex}}\hat{\mathbb{I}}(I, \Phi_{\mathrm{ex}})] + L\hat{J}_{\mathrm{c}}^{\oplus}$, where $\hat{\mathbb{I}}(I, \Phi_{\mathrm{ex}})$ is the identity operator on $\mathcal{H}(I, \Phi_{\mathrm{ex}})$;

(ii) in eigenvalue form $\Phi_{T_Q} = \Phi_{\mathrm{ex}} + Lj_Q^{\mathrm{c}}$.

(d) The 'flux quantization' condition: $\Phi_{T_Q} = -(\varphi_{\mathrm{l}} - \varphi_{\mathrm{r}})\Phi_0/2\pi$.

The maximum supercurrent I_{c} that can be passed through a dc SQUID is a Φ_0-periodic function of the external applied magnetic flux. Although this outcome is similar to the Aharonov–Bohm effect [24], the production of the interference pattern is not the same in the two cases. The Aharonov–Bohm effect, like most quantum (or, indeed, classical wave) interference patterns, is a result of phase differences between the two superposed wavefunctions, whereas in the dc SQUID, the interference effect is a result of the system having to satisfy so many constraints. In both cases, the extra phases picked up by the left- and right-hand wavefunctions are determined by the enclosed magnetic flux. In the Aharonov–Bohm effect, the enclosed flux is an independent external parameter, unaffected by the passing charged particles whose phases it affects, whereas in the dc SQUID, the enclosed flux is a function of the phase differences across both junctions. For the dc SQUID, the interference effect is not due to the left- and right-hand condensate wavefunctions being out of phase at their meeting point: they are not. It arises because the phases in the left- and right-hand paths are coupled by the JJs in each path, through the enclosed flux being a function of both. The interference effect is thus not the standard type of interference seen in quantum mechanical systems. The maximum through supercurrent $I_{\mathrm{c}}(\Phi_{\mathrm{ex}})$ marks the destruction of the superconducting state. Once exceeded, the system is driven normal and can then be described wholly in classical terms. $I_{\mathrm{c}}(\Phi_{\mathrm{ex}})$ thus defines the transition from the quantum to the classical realm. It is this boundary between the quantum and classical domains which exhibits the interference pattern.

4. Superselection rules and the quantum/classical divide

As we noted in the introduction, SSRs introduce classical properties into quantum systems and are thus closely linked to the distinction between quantum and classical systems. A commonly accepted definition of a classical system is that a system is called a classical system if all its observables commute [12]. According to this definition, our models of TSCR devices are classical systems. Yet physicists rightly recognize these devices as quantum systems: they are explicable only in quantum mechanical terms, despite their macroscopic size. In accepting these models as well as the quantum nature of superconducting rings, we are forced to give up the above definition and are left in the uncomfortable position of having no general kinematic criterion for ascertaining whether or not a given theoretical model refers to a quantum or a classical system. One possible solution to this dilemma is to seek a criterion in the dynamics of the system. In classical mechanics the evolution between disjoint states is given by classical canonical transformations [38, 39] which, unlike unitary evolutions between supersectors, do not forbid the Hamiltonian, which generates the canonical transformations, being an observable of the system.

It is useful here to make a distinction between *pure quantum systems*, which are those having no SSRs, so that there is a one-to-one correspondence between pure states and unit vectors, and *mixed quantum systems*, which possess a SSRs so that not all unit vectors represent pure states. Using this terminology, our models of TSCR systems, with an assumed unitary time evolution, are mixed quantum systems, rather than classical systems.

5. Use of mixed quantum systems as measuring devices

The use of a mixed quantum system such as a dc SQUID to measure a classical observable such as magnetic flux reverses the usual quantum measurement scenario in which a seemingly classical measuring device is used to measure a quantum observable, and might seem paradoxical. The crucial properties of a measuring device are, however, (1) that it has non-superposable macroscopically distinguishable states; and (2) that these may be unambiguously correlated with different states of the object being measured. The first of these properties means that a measuring device must possess a SSR, whatever type of system it is measuring: all measuring devices must be either mixed quantum systems or classical systems. The second requirement places a further constraint on measuring apparatus for quantum systems. This is because the measuring apparatus has to couple to the quantum system. The composite system comprising the measuring apparatus and quantum system is a mixed quantum system, so the coupling interaction should be described by a unitary time evolution. This means that the measuring apparatus itself must be a mixed quantum system, not a classical system.

In classical measurement theory, the correlation of object and apparatus states may be achieved with negligible effect on the state of the object. In principle, then, classical systems may be measured using apparatus that are either mixed quantum systems or purely classical ones. To measure a continuous classical observable to any arbitrary degree of accuracy, the measuring apparatus must have a continuous SSR, so that different states of the classical object may be put into one-to-one correspondence with different supersectors of the apparatus. Measuring apparatus for continuous classical observables may thus be either classical systems or mixed quantum systems possessing a continuous SSR. A dc SQUID is an example of the latter, with its pure states in one-to-one correspondence (over a one flux quantum range) with the classical external magnetic flux.

6. Incoherence and fluctuations in the environment

Thus far, we have assumed an ideal external current source. In practice, any current source will have fluctuations and uncertainty in the current produced. Suppose that a given non-ideal current source produces an output current with a distribution given by a Gaussian probability density function $w(I)$:

$$w(I) = \frac{1}{\sqrt{2\pi}\,\sigma}\,\mathrm{e}^{-(I-I_m)^2/2\sigma^2}. \tag{92}$$

That is, we have a current source generating an incoherent Gaussian output with a mean value I_m and an uncertainty σ. Likewise, the external magnetic flux arising from the applied magnetic field will have incoherent fluctuations. Clearly, if these are of the order of a flux quantum Φ_0 any interference effects will be washed out. Let us suppose that the external flux Φ_{ex} also has a Gaussian distribution given by a probability density function $\bar{w}(\Phi_{\mathrm{ex}})$. We have already established that an ideal current source feeding a precise current, say I_0, into the SQUID, together with an ideal magnetic field source giving rise to a precise flux, say Φ_{ex_0}, corresponds to a pure state $\Psi_{I_0,\Phi_{ex_0}}$ in the supersector $\mathcal{H}(I_0, \Phi_{ex_0})$. It follows that an incoherent current and magnetic flux correspond to a mixed state. Our model, in which various physical quantities are expressed explicitly in terms of direct integrals, is eminently suited to accommodate such a situation.

We can unify the description of pure and mixed states in terms of the statistical operators introduced in the appendix. The Hilbert space for the dc SQUID system is the direct integral of a two-parameter family of one-dimensional supersectors $\mathcal{H}(I, \Phi_{\mathrm{ex}})$. The pure state $\Psi_{I_0,\Phi_{ex_0}}$ is equivalent to the statistical operator $\hat{S}_p^{\oplus}$ on $\mathcal{H}^{\oplus}$:

$$\hat{S}_p^{\oplus} = \int^{\oplus} \mathrm{d}\Phi_{\mathrm{ex}} \int \hat{S}_p(I, \Phi_{\mathrm{ex}})\mathrm{d}I \tag{93}$$

where

$$\hat{S}_p(I, \Phi_{\mathrm{ex}}) = \delta(\Phi_{\mathrm{ex}} - \Phi_{ex_0})\delta(I - I_0)\hat{D}(I, \Phi_{\mathrm{ex}}) \qquad \hat{D}(I, \Phi_{\mathrm{ex}}) = |\Psi_{I,\Phi_{\mathrm{ex}}}\rangle\langle\Psi_{I,\Phi_{\mathrm{ex}}}|. \tag{94}$$

For incoherent currents and external fluxes, the mixed state of the system can be described by the statistical operator $\hat{S}_m^{\oplus}$:

$$\hat{S}_m^{\oplus} = \int^{\oplus} \mathrm{d}\Phi_{\mathrm{ex}} \int \hat{S}_m(I, \Phi_{\mathrm{ex}})\,\mathrm{d}I \qquad \text{where } \hat{S}_m(I, \Phi_{\mathrm{ex}}) = \bar{w}(\Phi_{\mathrm{ex}})w(I)\hat{D}(I, \Phi_{\mathrm{ex}}). \tag{95}$$

Expectation values of observables $\hat{A}^{\oplus}$ are given by $\mathrm{tr}(\hat{S}_m^{\oplus}\hat{A}^{\oplus})$. For example, the expectation value for the through current $\hat{J}_{\mathrm{t}}^{\oplus}$ is given by

$$\begin{aligned}\mathrm{tr}(\hat{S}_m^{\oplus}\hat{J}_{\mathrm{t}}^{\oplus}) &= \int \mathrm{d}\Phi_{\mathrm{ex}} \int \mathrm{tr}(\hat{S}_m(I, \Phi_{\mathrm{ex}})\hat{J}^{\mathrm{t}}(I, \Phi_{\mathrm{ex}}))\,\mathrm{d}I \\ &= \int \mathrm{d}\Phi_{\mathrm{ex}}\bar{w}(\Phi_{\mathrm{ex}}) \int w(I)\,\mathrm{tr}(\hat{D}(I, \Phi_{\mathrm{ex}})\hat{J}^{\mathrm{t}}(I, \Phi_{\mathrm{ex}}))\,\mathrm{d}I \\ &= \int w(I)\langle\Psi_{I,\Phi_{\mathrm{ex}}}|\hat{J}^{\mathrm{t}}(I, \Phi_{\mathrm{ex}})\Psi_{I,\Phi_{\mathrm{ex}}}\rangle\,\mathrm{d}I = I_m.\end{aligned} \tag{96}$$

In this manner we can incorporate the properties of both pure and mixed quantum systems into a unified model. The pure quantum part of the system is represented by pure states $\hat{S}_p^{\oplus}$, while the classical part corresponds to mixed states $\hat{S}_m^{\oplus}$.

7. Conclusions

Using the method of quantization by parts we have constructed exactly solvable models of two specific physical systems (namely a current-fed TSCR and a dc SQUID). This method has recently been used to develop a systematic treatment of branching quantum circuits [37, 40], showing how input and output leads could be incorporated into our present model to produce a complete model of the dc SQUID. All these systems have been modelled as quantum systems possessing continuous SSRs. The fact that they are explicable only by recourse to quantum mechanics, despite conforming to a widely used definition of a classical system, has led us to suggest that a re-examination of that definition should be undertaken. These models highlight the differences between standard quantum interference and the type of interference seen in SQUIDs. Consideration of the use of dc SQUIDs as measuring devices for classical fields has helped to identify the different attributes required of measuring apparatus in the quantum and classical domains.

Acknowledgments

FEH acknowledges support from the United Kingdom SERC, the Marsden Fund of the Royal Society of New Zealand and the University of Auckland Research Committee, and KKW thanks R H Fountain for valuable discussions.

Appendix: Continuous superselection rules, mixed states and statistical operators

In standard quantum mechanics, the quantum state of a system associated with a Hillbert space $\mathcal{H}$ may be represented by a density operator $\hat{D}$ on $\mathcal{H}$. In general, such a density operator represents a mixed state. A pure state, which can be described by a unit vector $\phi_0 \in \mathcal{H}$, corresponds to a density operator $\hat{D}_0 = |\phi_0\rangle\langle\phi_0|$ which is a projection operator on $\mathcal{H}$. Expectation values of observables $\hat{A}$ are given by $\mathrm{tr}(\hat{A}\hat{D})$, which is equal to $\langle\phi_0|\hat{A}\phi_0\rangle$ when $\hat{D} = \hat{D}_0$.

For a system associated with a Hilbert space with a continuous SSR the situation is technically quite different. The existence of a continuous SSR means, in the simple case where the SSR is parametrized by a single real parameter α, that:

(1) there exists a family of Hilbert spaces $\mathcal{H}(\alpha)$ parametrized by α such that the Hilbert space for the system is given by the direct integral space

$$\mathcal{H}^{\oplus} = \int^{\oplus} \mathcal{H}(\alpha)\, \mathrm{d}\alpha. \tag{97}$$

A unit vector $\phi^{\oplus}$ in the direct integral space $\mathcal{H}^{\oplus}$ is of the form

$$\phi^{\oplus} = \int^{\oplus} c(\alpha)\phi(\alpha)\, \mathrm{d}\alpha \tag{98}$$

where $\phi(\alpha)$ are unit vectors in $\mathcal{H}(\alpha)$ and $c(\alpha)$ is a complex function of α such that $\int |c(\alpha)|^2\, \mathrm{d}\alpha = 1$.

(2) Physical observables for the system correspond to self-adjoint operators in $\mathcal{H}^{\oplus}$ of the form

$$\hat{A}^{\oplus} = \int^{\oplus} \hat{A}(\alpha)\, \mathrm{d}\alpha \tag{99}$$

where $\hat{A}(\alpha)$ are self-adjoint operators in $\mathcal{H}(\alpha)$. Operators of this type are called decomposable self-adjoint operators in $\mathcal{H}^{\oplus}$ [22, 23].

(3) A unit vector $\phi^{\oplus}$ generally represents a mixture of states $\phi(\alpha) \in \mathcal{H}(\alpha)$, because of the decomposable nature of the operators representing observables. The expectation value of $\hat{A}^{\oplus}$ is

$$\langle\phi^{\oplus}|\hat{A}^{\oplus}\phi^{\oplus}\rangle = \int |c(\alpha)|^2 \langle\phi(\alpha)|\hat{A}(\alpha)\phi(\alpha)\rangle\, \mathrm{d}\alpha \tag{100}$$

where $\langle\phi(\alpha)|\hat{A}(\alpha)\phi(\alpha)\rangle$ is the expectation value of $\hat{A}(\alpha)$ with respect to $\phi(\alpha)$ in the supersector $\mathcal{H}(\alpha)$. Note that this mixture is quite distinct from the mixed state described by a density operator $\hat{D}$ in the Hilbert space $\mathcal{H}$ in standard quantum mechanics. The above mixture is open to the ignorance interpretation and represents a classical property.

We can generalize the description of mixed states by introducing statistical operators $\hat{S}^{\oplus}$, defined on the direct integral space $\mathcal{H}^{\oplus}$ by

$$\hat{S}^{\oplus} = \int^{\oplus} \hat{S}(\alpha)\,\mathrm{d}\alpha = \int^{\oplus} w(\alpha)\hat{D}(\alpha)\,\mathrm{d}\alpha \qquad \hat{S}(\alpha) = w(\alpha)\hat{D}(\alpha) \tag{101}$$

where $\hat{D}(\alpha)$ is a density operator on $\mathcal{H}(\alpha)$ and $w(\alpha)$ is a real-valued probability density function (i.e. $0 \leqslant w(\alpha) \leqslant 1$ and $\int w(\alpha)\,\mathrm{d}\alpha = 1$).

In the case where all supersectors $\mathcal{H}(\alpha)$ are one dimensional, $\hat{D}(\alpha) = |\phi(\alpha)\rangle\langle\phi(\alpha)|$ and we can see that $\hat{S}^{\oplus}$ corresponds to a mixture of states $\phi(\alpha)$ with the distribution of states in the mixture given by the probability density function $w(\alpha)$. In other words, this statistical operator can be generated by any unit vector $\phi^{\oplus}$ for which $|c(\alpha)|^2 = w(\alpha)$. Consequently, a statistical operator does not correspond to a unique vector in $\mathcal{H}^{\oplus}$ [1, 4]. We can represent a pure state $\phi(\alpha_0) \in \mathcal{H}(\alpha_0)$ in terms of a statistical operator with a δ-function distribution $w(\alpha) = \delta(\alpha - \alpha_0)$:

$$\hat{S}_p^{\oplus} = \int^{\oplus} \hat{S}_p(\alpha)\,\mathrm{d}\alpha = \int^{\oplus} \delta(\alpha - \alpha_0)\hat{D}(\alpha)\,\mathrm{d}\alpha = |\phi(\alpha_0)\rangle\langle\phi(\alpha_0)|. \tag{102}$$

Expectation values of observables are given by

$$\mathrm{tr}(\hat{S}^{\oplus}\hat{A}^{\oplus}) = \int \mathrm{tr}(\hat{A}(\alpha)\hat{S}(\alpha))\,\mathrm{d}\alpha \tag{103}$$

where the trace $\mathrm{tr}(\hat{A}(\alpha)\hat{S}(\alpha))$ is evaluated in the supersector $\mathcal{H}(\alpha)$. We should point out that a statistical operator $\hat{S}^{\oplus}$ as defined above is generally not a density operator on $\mathcal{H}^{\oplus}$, and the converse is also true (for example, a density operator need not be decomposable). This is in distinction to some other definitions of statistical operators [41]. All these definitions can be extended to the case where the supersectors are specified by two or more real parameters.

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